Method of moments for the dilute granular flow of inelastic spheres

Matteo Strumendo and Paolo Canu*

DIPIC—Dipartimento di Principi e Impianti di Ingegneria Chimica, Università di Padova, 35131 Padova, Italy

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Some peculiar features of granular materials (smooth, identical spheres) in rapid flow are the normal pressure differences and the related anisotropy of the velocity distribution function $f^{(1)}$. Kinetic theories have been proposed that account for the anisotropy, mostly based on a generalization of the Chapman-Enskog expansion [N. Sela and I. Goldhirsch, J. Fluid Mech. **361**, 41 (1998)]. In the present paper, we approach the problem differently by means of the method of moments; previously, similar theories have been constructed for the nearly elastic behavior of granular matter but were not able to predict the normal pressures differences. To overcome these restrictions, we use as an approximation of the $f^{(1)}$ a truncated series expansion in Hermite polynomials around the Maxwellian distribution function. We used the approximated $f^{(1)}$ to evaluate the collisional source term and calculated all the resulting integrals; also, the difference in the mean velocity of the two colliding particles has been taken into account. To simulate the granular flows, all the second-order moment balances are considered together with the mass and momentum balances. In balance equations of the Nth-order moments, the $(N+1)$th-order moments (and their derivatives) appear: we therefore introduced closure equations to express them as functions of lower-order moments by a generalization of the ''elementary kinetic theory,'' instead of the classical procedure of neglecting the $(N+1)$th-order moments and their derivatives. We applied the model to the translational flow on an inclined chute obtaining the profiles of the solid volumetric fraction, the mean velocity, and all the second-order moments. The theoretical results have been compared with experimental data [E. Azanza, F. Chevoir, and P. Moucheront, J. Fluid Mech. **400**, 199 (1999); T. G. Drake, J. Fluid Mech. **225**, 121 (1991)] and all the features of the flow are reflected by the model: the decreasing exponential profile of the solid volumetric fraction, the parabolic shape of the mean velocity, the constancy of the granular temperature and of its components. Besides, the model predicts the normal pressures differences, typical of the granular materials.

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I. INTRODUCTION

In recent years, many efforts have been made to obtain a better understanding of the granular matter, owing to the large number of industrial processes that involve granular materials. In the literature [1,2], the regimes of flow of a granular material have been classified as frictional, collisional, translational, and viscous flow. Theoretical work on the dynamics of granular matter reflects the classification of the regimes. For the frictional regime, a basic idea has been to extend to the granular state the Coulomb friction law. Sokolovskij [3] introduced the Mohr-Coulomb criterion in the continuum (static) balances of the forces for a yielding granular material; others [4,5] generalized to a granular material concepts of the theory of plasticity (yielding criteria, particularly the Mohr-Coulomb criterion, and the flow rules) in order to describe the transition from static to dynamic behavior and the slow movement of the material. These (or similar) models have been used subsequently to simulate the flow on inclined chutes [6,7] or the shear flow [8]. Another approach was carried on by Cowin [9–14] (but also Jenkins [15]) and was based on continuum thermodynamic considerations [16,17] and on the analogy with models of liquid crystals and polar fluids [18,19]; his work resulted in the introduction of the density gradient in the constitutive relations of the granular material. A point of view close to that of Cowin was developed by Kanatani [20] who applied methods taken from the theories of micropolar fluids [21–26] (and, more recently, Ref. [27]) to the flow of granular media; he also tried to take into account the effect of the couple stress [28–34], writing a balance equation for the angular momentum too. Different, more empirical approaches can be found in the works of Pouliquen and Gutfraind [35], in which a probabilistic model is coupled with the Coulomb relation, and Santomaso and Canu [36], who considered the granular media as a pseudofluid that follows a non-Newtonian behavior.

On the other side there are works devoted to the rapid flow (collisional and translational regimes). This area has been investigated essentially by means of computer simulations [37,38] or by a microstructural kinetic modeling. The basic ideas of this second approach (to which the present paper belongs) were first presented in the pioneering work of Bagnold [1], who constructed a simplified expression to relate the gradient of the mean velocity with the stresses, and above all by Ogawa [39,40], who emphasized the presence of the fluctuating components of the velocity of the particles and introduced the concept of ''granular temperature,'' stating a clear analogy between the kinetic theory of gases (dilute and dense) and the behavior of granular media (translational and collisional regimes). Following Ogawa, Haff [41] wrote a paper in which he, with heuristic arguments, determined the transport coefficients and the collisional dissipation of mechanical energy and found the solutions of the

*Corresponding author. Mailing address: DIPIC—Dipartimento di Principi e Impianti di Ingegneria Chimica, Università di Padova, Via Marzolo 9, 35131, Padova, Italy. FAX: (+39) 049 8275461. Email address: paolo.canu@unipd.it

balance equations for some common applications. Then, researchers went deeper inside the analogy with the kinetic theory of gases [42–44] and introduced in the granular field the use of the distribution functions. First, for a system comprizing smooth, rigid, elastic spheres, a Maxwellian distribution function was used [45], then [2,46,47] balance equations in a general form and expressions for collisional source and flux terms were given and applications were made to smooth, nearly elastic spheres using non-Maxwellian distribution functions. Attempts to generalize these ideas have been made to include rough particles [48] and the case of a mixture of particles [49].

Simulations of the granular behavior taking into account all the regimes (frictional, collisional, translational) have been performed by Savage [50,6], Johnson, Nott, and Jackson [7], Anderson and Jackson [51] in the case of the inclined chute and by Johnson and Jackson [8] in the case of plane shearing. Simulations of the chute flow limited to the rapid regimes have been proposed by Richman and Marciniec [52] and recently by Massoudi and Boyle [53]. The problem of the vertical vibration of granular material has been approached, among the others, by Warr, Huntley, and Jacques [54].

The presence of the viscous contribution to the transfer of particles properties has been considered by Nott and Jackson [55] while the influence of the boundary conditions has been investigated by Hue *et al.* [56], Jenkins and Richman [57], and recently by Chou [58].

Turning back to the case of smooth, spherical, identical particles in rapid flow, it has been recently pointed out [59–61] that, while for a gas the distribution function at equilibrium is Maxwellian and in some applications the Maxwellian approximation is satisfactory, granular matter also in the simplest cases deviates from the Maxwellian behavior. Accounting for deviations, through all second-order moments, for example, is required to give a proper description of the granular dynamics and to represent phenomena that are negligible in the classical fluids but not in the granular field, as the differences between the normal pressures. Richman [59] introduced an anisotropic Maxwell distribution function, dependent on all second-order moments of the fluctuant velocities, and used all the second-order moment balances to solve the dilute, steady, homogeneous shear flows. A systematic approach has been developed by Goldhirsch and Sela [60,61], based on a generalization of the Chapman-Enskog expansion. A direct analysis of the Boltzmann equation via an appropriate Chapman-Enskog expansion was previously suggested by Goldshtein and Shapiro [62], who studied a weakly inhomogeneous system consisting of rough spheres and calculated the partition of fluctuating kinetic energy between the rotational and translational components. Sela and Goldhirsch [61] proposed a method to generalize the Chapman-Enskog expansion to smooth inelastic spheres based on a double expansion with respect to both the Knudsen number and the degree of inelasticity. They obtained constitutive relations for the heat flux and for the stress tensor and calculated the normal pressure difference in the case of the shear flow, resulting in a good agreement with the numerical calculations. A shortcoming of all the methods based on the (generalized) Chapman-Enskog expansion is that they can predict only normal solutions and not the most general ones [42].

In the present work, we apply the method of moments to predict the behavior of fast flow of smooth, identical spheres and particularly to account for the anisotropy of the second-order moments. The principal features of this paper are the following. First, beyond the mass and momentum balances, all the second-order moment balances are considered. Second, the velocity distribution function $f^{(1)}$ has been approximated by a truncated series expansion in Hermite polynomials around the Maxwellian distribution function [63] in a fashion similar to that of Jenkins and Richman [47]. The integrals of the collisional source term are rigorously evaluated, without introducing further restrictions corresponding to the hypothesis of nearly elastic particles; also the difference in the mean velocity of the two colliding particles is considered in the evaluation of the collisional source term. Finally, having used the method of moments [64], the closure equations have been specified: the $(N+1)$th order moments (N being the order of the moments of which we consider the balances) and their derivatives have been expressed in terms of the lower-order moments introducing some improvements to the classical "elementary kinetic theory" [65,42]. This choice is different with respect to the solution adopted by Grad [63] and by Jenkins and Richman [47], who considered equal to zero the $(N+1)$th-order moments and their derivatives, in the hypothesis that the $f^{(1)}$ is approximated by an Hermite expansion truncated to the Nth order. The model has been applied to the translational flow on an inclined chute, obtaining the profiles of the solid volumetric fraction, the mean velocity, and all the second-order moments. The model can be extended to all the rapid regimes once the collisional flux terms are considered; expressions for these fluxes have been given by Jenkins and Savage [2] and by Jenkins and Richman [47]. Finally, the theoretical results have been compared with the experimental data of Azanza, Chevoir, and Moucheront [66] and of Drake [67]. Other measurements have been presented by Savage [50,68], Ishida and Shirai [69], Ahn, Brennen, and Sabersky [70], Hanes and Walton [38], Santomaso and Canu [36] but the papers of Azanza, Chevoir, and Moucheront [66] and of Drake [67] report the most complete series of data, including the profiles of second-order moments and the values of the microstructural properties and of the operative parameters (restitution coefficient, chute angle, particles diameter, and density).

II. STATEMENT OF THE PROBLEM AND DEFINITIONS

We will consider particles that are smooth spheres of uniform diameter D; the particles are noncohesive and electrostatic effects are neglected. The particle interactions are only binary instantaneous collisions.

The single particle velocity distribution function $f^{(1)}$ is defined so that $f^{(1)}(\mathbf{c},\mathbf{r},t)d\mathbf{c}\,d\mathbf{r}$ is the probable number of particles with actual velocities in $\mathbf{c}+d\mathbf{c}$, in the volume element $\mathbf{r}+d\mathbf{r}$, at the time t. The actual particle velocity $\mathbf{c}$ can be assumed as the sum of the fluctuant velocity $\mathbf{C}$ and the mean velocity $\mathbf{u}$. the single particle velocity distribution function can be expressed both as a function of $(\mathbf{c},\mathbf{r},t)$ or as a function of the fluctuant velocity $(\mathbf{C},\mathbf{r},t)$ and

$$f^{(1)}(\mathbf{c},\mathbf{r},t)d\mathbf{c}\,d\mathbf{r}=f^{(1)}(\mathbf{C}+\mathbf{u}(\mathbf{r},t),\mathbf{r},t)d\mathbf{C}\,d\mathbf{r}$$
$$=f_c^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}\,d\mathbf{r}. \quad (1)$$

The zeroth-order moment is defined by $\int f_c^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}$ and is equal to the number density $n(\mathbf{r},t)$ (number of particles per unit of volume); here and in the following $d\mathbf{C}=dC_x\,dC_y\,dC_z$ and the integration is intended over all values of $\mathbf{C}$. The first-order moments of the fluctuant velocities are defined by $\int C_i f_c^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}$ and are always equal to zero; the second-order moments of the fluctuant velocities are given by $\int C_i C_j f_c^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}$, where the generic indices i and j refer to the ith and jth components of the fluctuant velocity. Similarly, it is possible to define the fluctuant velocity moments of higher order.

In the same manner, actual velocity moments of a generic order can be defined; the first-order moments of the actual velocities are $\int c_i f^{(1)}(\mathbf{c},\mathbf{r},t)d\mathbf{c}$ and they are equal to the ith component of the mean velocity u_i multiplied by n. The mean value of a generic property which depends upon the actual velocity $\psi(\mathbf{c})$ is given by $\langle\psi\rangle\equiv(n^{-1})\int\psi(\mathbf{c})f^{(1)}(\mathbf{c},\mathbf{r},t)d\mathbf{c}$; or, in terms of fluctuations, if $\psi=\psi(\mathbf{C})$ $\langle\psi\rangle\equiv(n^{-1})\int\psi(\mathbf{C})f_c^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}$. Following Jenkins and Richman [47], we introduce the notation $M_{i_1,i_2\ldots i_N}\equiv\langle C_{i_1}C_{i_2}\ldots C_{i_N}\rangle$ for the generic Nth-order moment of the fluctuating velocities divided by n; the granular temperature is defined by $T=(M_{xx}+M_{yy}+M_{zz})/3$ and represents a measure of the kinetic energy of the particles associated with the fluctuating velocities, which is the only "internal" energy to be considered for smooth particles.

The pair distribution function $f^{(2)}$ is defined so that $f^{(2)}(\mathbf{c}_1,\mathbf{r}_1,\mathbf{c}_2,\mathbf{r}_2,t)d\mathbf{c}_1\,d\mathbf{r}_1\,d\mathbf{c}_2\,d\mathbf{r}_2$ is the joint probability that at the same time t a particle with actual velocity $(\mathbf{c}_1+d\mathbf{c}_1)$ is in $(\mathbf{r}_1+d\mathbf{r}_1)$ and a particle with actual velocity $(\mathbf{c}_2+d\mathbf{c}_2)$ is in $(\mathbf{r}_2+d\mathbf{r}_2)$. For granular materials in translational regime, the hypothesis of "molecular chaos" can be used, according to which the two particles move independently,

$$f^{(2)}(\mathbf{c}_1,\mathbf{r}_1,\mathbf{c}_2,\mathbf{r}_2,t)=f^{(1)}(\mathbf{c}_1,\mathbf{r}_1,t)f^{(1)}(\mathbf{c}_2,\mathbf{r}_2,t), \quad (2)$$

if we are considering moderately dense systems (collisional regime), in which a significant fraction of the volume is occupied by the particles, Eq. (2) must be corrected into

$$f^{(2)}(\mathbf{c}_1,\mathbf{r}_1,\mathbf{c}_2,\mathbf{r}_1+D\mathbf{k},t)$$
$$=g_0(\mathbf{r}_1+\tfrac{1}{2}D\mathbf{k},t)f^{(1)}(\mathbf{c}_1,\mathbf{r}_1,t)f^{(1)}(\mathbf{c}_2,\mathbf{r}_1+D\mathbf{k},t), \quad (3)$$

in the case of $\mathbf{r}_2=\mathbf{r}_1+D\mathbf{k}$, where $\mathbf{k}$ is the unit vector directed from the center of the particle in $\mathbf{r}_1$ to the particle in $\mathbf{r}_2$, as depicted in Fig. 1. The function g_0 is the radial distribution function at contact and it has been introduced in Eq. (3) to reflect the increase of collisions in moderately dense systems, where a fraction of the volume is occupied by the particles [42,71,70].

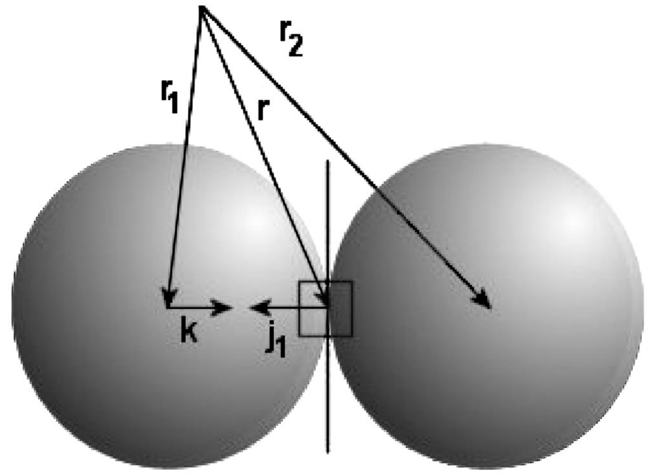

FIG. 1. Definitions of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}$, position of the particle's centers and of the contact point, respectively. $\mathbf{r}_1=\mathbf{r}+\mathbf{j}_1(D/2)$. $\mathbf{k}$ is the unit vector directed from the center of the first particle to the center of the second.

III. THE APPROXIMATION FOR $f_C^{(1)}$

To develop an explicit expression for $f_C^{(1)}$ we follow a method similar to that suggested by Grad [63] in the theory of dilute gases, i.e., we use a series expansion based on the Maxwell distribution function,

$$f_C^{(1)}(\mathbf{C},\mathbf{r},t)=f_{C,0}^{(1)}(\mathbf{C},\mathbf{r},t)\sum_{n=0}^{\infty}a_{i_n}^{(n)}(\mathbf{r},t)H_{i_n}^{(n)}(\mathbf{w}), \quad (4)$$

where $f_{C,0}^{(1)}(\mathbf{C},\mathbf{r},t)=n/(2\pi T)^{3/2}\exp(-|\mathbf{C}|^2/2T)$ is the Maxwell distribution function, $\mathbf{w}=\mathbf{C}/\sqrt{T}$ is the nondimensional form of $\mathbf{C}$ and i_n is a permutation of n indices chosen among x,y,z. The summation is intended over n and, for each value of n, over all the permutations of n indices (chosen among x,y,z). $a_{i_n}^{(n)}$ are the expansion coefficients and the functions $H_{i_n}^{(n)}$ are Hermite orthogonal polynomials [63].

The polynomials $H_{i_n}^{(n)}$ are symmetrical with respect to every combination of their indices, that is, $H_{i_n}^{(n)}=H_{j_n}^{(n)}$ if j_n is a permutation of i_n and the related coefficients $a_{i_n}^{(n)}$ and $a_{j_n}^{(n)}$ are equal too; so, the same quantity can be written more times in the expansion. Instead, we prefer to introduce a single coefficient $b_{j_n}^{(n)}$, equal to the summation $(a_{i_n}^{(n)}+a_{j_n}^{(n)}+\cdots)$ of the $a_{i_n}^{(n)}$ over the different permutations i_n for a fixed choice of the indices; all these $a_{i_n}^{(n)}$ to be summed are identical Accordingly, the expansion (4) becomes

$$f_C^{(1)}=f_{C,0}^{(1)}\sum_{n=0}^{\infty}b_{i_n}^{(n)}H_{i_n}^{(n)}, \quad (5)$$

where the summation is intended over n and, for each value of n, over the choices of the indices x,y,z (but not over their permutations).

The expressions of the $b_{i_n}^{(n)}$ can be obtained requiring the condition of orthogonality of $H_{i_n}^{(n)}$ using $f_{C,0}^{(1)}$ as weight function. Up to the third order they result in

$$b^0=1, \tag{6}$$

$$b_i^{(1)}=0, \tag{7}$$

$$(\beta)_{ij}^{(2)} b_{ij}^{(2)}=\frac{M_{ij}}{T}-\delta_{ij}, \tag{8}$$

$$(\beta)_{ijk}^{(3)} b_{ijk}^{(3)}=\frac{M_{ijk}}{T^{3/2}}, \tag{9}$$

with $(\beta)_{ij}^{(2)}=2$ if $i=j$ $(\beta)_{ij}^{(2)}=1$ if $i\neq j$, $(\beta)_{ijk}^{(3)}=6$ if $i=j=k$ $(\beta)_{ijk}^{(3)}=2$ if two of the three indices are equal $(\beta)_{ijk}^{(3)}=1$ if all the three indices are different. It can be easily seen that the sum $b_{xx}+b_{yy}+b_{zz}=0$.

IV. BALANCE EQUATIONS

Balance equations can be written following Chapman and Cowling [42]. They derived the balance for a generic property ψ function of t, $\mathbf{r}$, $\mathbf{c}$ from the Boltzmann equation,

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{r},\mathbf{c}} n\langle\psi\rangle = n\frac{F_i}{m}\left\langle \left.\frac{\delta\psi}{\partial c_i}\right|_{t,\mathbf{r}}\right\rangle - \left.\frac{\partial n\langle c_i,\psi\rangle}{\partial r_i}\right|_{t,\mathbf{c}} + n\left(\left\langle \left.\frac{\partial\psi}{\partial t}\right|_{\mathbf{r},\mathbf{c}}\right\rangle + \left\langle c_i \left.\frac{\partial\psi}{\partial r_i}\right|_{t,\mathbf{c}}\right\rangle\right)+\mathcal{C}(\psi), \tag{10}$$

where F_i are the external forces and m is the mass of a particle. The summation over the same indices convention has been adopted and so will be in the following. $\mathcal{C}(\psi)$ is the rate of change of the property ψ per unit volume due to collisions.

In the present work we will use properties $\psi=\psi(\mathbf{C})$ depending only on the fluctuating velocities, which do not depend on t and on $\mathbf{r}$; in this hypothesis and if ψ is expressed as a function of $\mathbf{C}$ (fluctuating velocities) instead of $\mathbf{c}$, the balance (10) can be rewritten as [47]

$$\frac{D\rho\langle\psi\rangle}{Dt}+\rho\langle\psi\rangle\left.\frac{\partial u_i}{\partial r_i}\right|_{t,\mathbf{C}}+\left.\frac{\partial}{\partial r_i}\right|_{t,\mathbf{C}}[\rho\langle C_i\psi\rangle+\Theta_i(m\psi)] + \rho\left(\frac{Du_i}{Dt}-\frac{F_i}{m}\right)\left\langle \left.\frac{\partial\psi}{\partial C_i}\right|_{t,\mathbf{r}}\right\rangle + \left(\rho\left\langle C_i \left.\frac{\partial\psi}{\partial C_j}\right|_{t,\mathbf{r}}\right\rangle + \Theta_i\left(\left.\frac{\partial m\psi}{\partial C_j}\right|_{t,\mathbf{r}}\right)\right)\left.\frac{\partial u_j}{\partial r_i}\right|_{t,\mathbf{C}} = \chi(m\psi), \tag{11}$$

in which

$$\frac{D}{Dt}=\left.\frac{\partial}{\partial t}\right|_{\mathbf{r},\mathbf{C}}+u_i\left.\frac{\partial}{\partial r_i}\right|_{t,\mathbf{C}},$$

$\rho=nm$ is the mass bulk density, u_i is the mean velocity, $\Theta_i(\psi)$ is the collisional flux of the property ψ in the direction i, and $\chi(\psi)$ is the collisional source of the property ψ in the unit of time and of volume.

From the general population balance (11), it is possible to derive specific balances depending on the type of function $\psi(\mathbf{C})$. In the following, the balance of mass [Eq. (12)], the three momentum balances [Eq. (13)] the balance of energy [Eq. (14)], and of the deviatoric part of the second-order moments of the fluctuant velocity [Eq. (15)] are reported in the form of Jenkins and Richman [47],

$$\frac{D\rho}{Dt}+\rho\frac{\partial u_i}{\partial r_i}=0, \tag{12}$$

$$\rho\frac{Du_i}{Dt}+\frac{\partial P_{ij}}{\partial r_j}=nF_i, \tag{13}$$

$$\frac{3}{2}\rho\frac{DT}{Dt}+\frac{\partial Q_i}{\partial r_i}+P_{ij}\frac{\partial u_j}{\partial r_l}=\frac{1}{2}\sum_{j=x,y,z}\chi_{ij}, \tag{14}$$

$$\tfrac{1}{2}\rho\dot{\hat{M}}_{ij}+(Q_{kij}-\tfrac{1}{3}Q_k\delta_{ij})_{,k}+\tfrac{1}{2}[P_{ki}u_{j,k}+P_{kj}u_{i,k}] - \tfrac{1}{3}P_{kn}u_{n,k}\delta_{ij}=\tfrac{1}{2}\hat{\chi}_{ij}, \tag{15}$$

in which

$$\chi_{i_1 i_2\cdots i_N}\equiv\chi(mC_{i_1}C_{i_2}\cdots C_{i_N}),$$

$$\Theta_{j i_1 i_2\cdots i_N}\equiv\Theta_j(mC_{i_1}C_{i_2}\cdots C_{i_N}),$$

$$P_{ij}=\rho M_{ij}+\Theta_{ij},$$

equal to the sum of the diffusive translational momentum flux [42] and the collisional flux of the property mC_j; $Q_i=1/2\Sigma_{j=x,y,z}(\rho M_{ijj}+\Theta_{ijj})$ and $\Sigma_{j=x,y,z}(\rho M_{ijj}+\Theta_{ijj})$ is equal to the sum of the diffusive translational flux of energy and the collisional flux of energy along the ith direction; $Q_{kij}=(\rho M_{kij}+\Theta_{kij})/2$,

$$\hat{M}_{ij}=M_{ij}-\sum_{k=x,y,z}(M_{kk}/3)\delta_{ij},$$

$$\hat{\chi}_{ij}=\chi_{ij}-\sum_{k=x,y,z}(\chi_{kk}/3)\delta_{ij},$$

$$\dot{\hat{M}}_{ij}=\frac{D\hat{M}_{ij}}{Dt}.$$

Similarly, higher-order moment balances can be derived. Note that the first-order moment balance, Eq. (13), contains M_{ij}, i.e., second-order moments; similarly the second-order moment balances, Eqs. (14) and (15), contain third-order moments, and so on. Solutions of these equations require some sort of closure, discussed in Sec. VII.

V. THE COLLISIONAL SOURCE OF A PROPERTY

To solve the balance equations [see Eq. (11) for a general expression], we need an expression for the collisional contributions $\chi(\psi)$ and $\Theta_i(\psi)$. In the following we will develop an expression for $\chi(\psi)$ while we will not discuss the collisional fluxes Θ_i, already determined by Jenkins and Savage [2] and Jenkins and Richman [47].

If we consider two particles labeled 1 and 2, we define ψ_1 as the property ψ of particle 1 and ψ_2 as the same property for particle 2. The meaning of $\chi(\psi)$ is the variation of the sum $(\psi_1+\psi_2)$ due to the collisions between two particles, that occur inside the element of volume $d\mathbf{r}$ around the reference point $\mathbf{r}$ and during the time $t\rightarrow t+dt$, per unit of volume $d\mathbf{r}$ and of time dt. The collisions are considered instantaneous. When a collision takes place, the variation in the sum $(\psi_1+\psi_2)$ will be associated with the volume element $d\mathbf{r}$ where the contact point lies (Fig. 1). To calculate $\chi(\psi)$ we have to know the probability that a collision might take place between two particles in $(\mathbf{r},d\mathbf{r})$ during (t,dt) and we call it P_{coll}.

The function $f_c^{(1)}$ is defined so that $f_c^{(1)}(\mathbf{c},\mathbf{r},t)d\mathbf{c}\,d\mathbf{r}$ represents the probable number of particles with actual velocities in $\mathbf{c}+d\mathbf{c}$, whose center is inside the volume element $\mathbf{r}+d\mathbf{r}$, at the time t. It can be demonstrated that $f_c^{(1)}\cong f^{(1)}$.

The probability P_{coll} that a collision might take place between two particles in $(\mathbf{r},d\mathbf{r})$ during (t,dt) is given by the joint probability of two events to happen simultaneously. The first event is that the particle 1, candidate for collision in (t,dt), at the time t is in $\mathbf{r}_1$, $d\mathbf{r}_1$ with $\mathbf{r}_1=\mathbf{r}+(D/2)\mathbf{j}_1$, where $\mathbf{j}_1$ is a generic unit vector applied in $\mathbf{r}$ (Fig. 1). The second event is that at the time t there are other particles (particles 2), with actual velocities around $\mathbf{c}_2+d\mathbf{c}_2$, likely to collide against particle 1 in the following dt and the contact point between the particles at impact would be in $\mathbf{r}$, $d\mathbf{r}$. The joint probability of both events can be determined once each single event's probability is given.

Let us consider in $\mathbf{r}$ a rectangular reference frame x,y,z and a spherical reference frame ϕ, θ, R where R is the radial coordinate, θ is the angle between $R\mathbf{j}_1$ and y, and φ is the angle between the projection of $R\mathbf{j}_1$ on the x-z plane and x. Points $\mathbf{r}_1$ can be approximately described by the spherical shell between two spheres centred in $\mathbf{r}$, the first with radius $(D-dR)/2$, the second with radius $(D+dR)/2$.

The probable number of particles with actual velocities around $\mathbf{c}_1+d\mathbf{c}_1$ whose center is in the volume element $(D/2)^2dR\,d\mathbf{j}_1$ around $\mathbf{r}_1$ (Fig. 2), at the time t (first event), is

$$f_c^{(1)}(\mathbf{c}_1,\mathbf{r}_1,t)d\mathbf{c}_1\left(\frac{D}{2}\right)^2 dR\,d\mathbf{j}_1 \cong f^{(1)}(\mathbf{c}_1,\mathbf{r}_1,t)d\mathbf{c}_1\left(\frac{D}{2}\right)^2 dR\,d\mathbf{j}_1. \tag{16}$$

To determine the probability of the second event let us consider a particle 1 centered in the point $\mathbf{r}_1=\mathbf{r}+(D/2)\cdot\mathbf{j}_1$. If we connect the center of particle 2 with the center of particle 1 by a straight line defined by a unit vector $\mathbf{k}$ (Fig. 1), only those particles 2 situated at the time of impact in points along

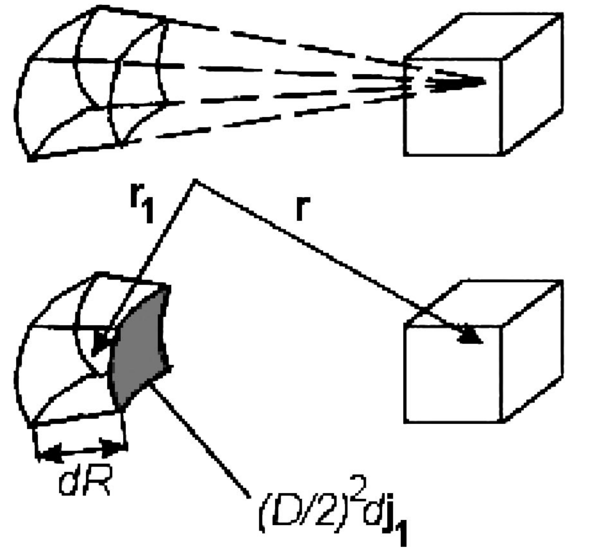

FIG. 2. Definition of the volume element around $\mathbf{r}_1$ in a spherical reference frame. R is the radial coordinate; $d\mathbf{j}_1$ is the differential of the solid angle and defines the surface element $(D/2)^2d\mathbf{j}_1$ on the sphere of radius $D/2$ centered in $\mathbf{r}$.

the lines with $\mathbf{k}$ inside the interval shown in Fig. 3 have to be considered, otherwise the lines will not pass across the volume $\mathbf{r}$, $d\mathbf{r}$ and the collision that we want to monitor will not occur in $\mathbf{r}$, $d\mathbf{r}$. So, the second event is that there are particles 2, with actual velocities around $\mathbf{c}_2+d\mathbf{c}_2$, the center of which can be connected at the impact to the center of particle 1 by a unit vector $\mathbf{k}$, with $\mathbf{k}$ inside the interval shown in Fig. 3.

Let us consider a volume element around a point $\mathbf{r}_2$ (Fig. 4, where a geometrical sphere of radius $\underline{R}$, not a particle, is shown) whose base is the surface element $\underline{R}^2d\mathbf{j}$ and generatrix $d\mathbf{l}$. The probable number of particles with actual velocities around $\mathbf{c}_2+d\mathbf{c}_2$ whose center is in a volume element $\mathbf{r}_2+d\mathbf{r}_2$, at the time t, is

$$f^{(1)}(\mathbf{c}_2,\mathbf{r}_2,t)dc_2\underline{R}^2d\mathbf{j}(d\mathbf{l}\cdot\mathbf{j}).$$

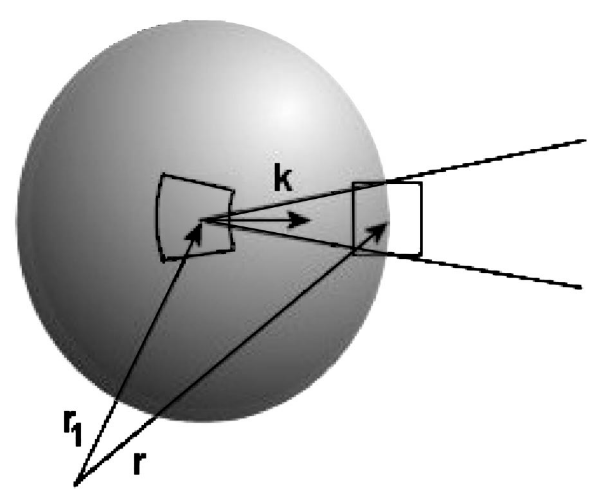

FIG. 3. Range of $\mathbf{k}$ such that the impact between the particles is inside $(\mathbf{r},d\mathbf{r})$.

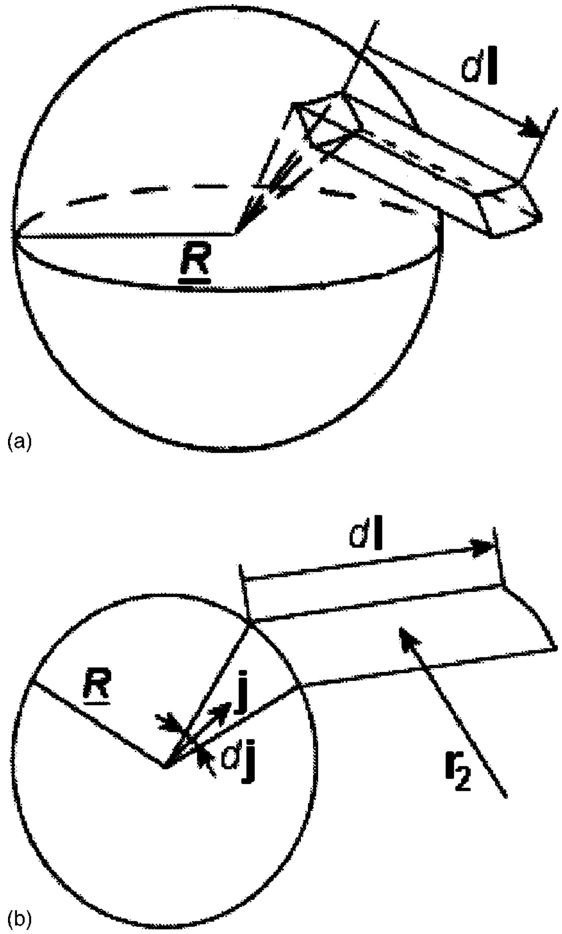

FIG. 4. Definition of the volume element around the point $\mathbf{r}_2$ by means of the generatrix $d\mathbf{l}$ and the base (surface element) $\underline{R}^2\,d\mathbf{j}$; $\mathbf{j}$ is a unit vector and $\underline{R}$ is the radius of a geometrical sphere, not of a particle. The same image is shown in three-dimensional (a) and two-dimensional (b) views.

The direction of $d\mathbf{l}$ is such that $(d\mathbf{l}\cdot\mathbf{j})>0$.

The particles 2, fulfilling the requirements above, at t_{impact} lay on the element surface dS of the geometrical sphere shown in Fig. 5, which approximately measures $4dA$, where dA is the magnitude of any of the six faces that are the boundaries of the volume element $d\mathbf{r}$ ($dA=dz\,dy$ in Fig. 5). These particles in the time t before t_{impact} are inside the volume whose base measures $4dA$ and whose generatrix is $(\mathbf{c}_1-\mathbf{c}_2)dt$ [42]. Therefore the probability of the second event is given by

$$f^{(1)}(\mathbf{c}_1,\mathbf{r}_2,t)d\mathbf{c}_2\{[(\mathbf{c}_1-\mathbf{c}_2)dt]\cdot\mathbf{k}\}4dA,$$

$$\mathbf{r}_2=r+\frac{D}{2}\mathbf{k}+\frac{1}{2}(\mathbf{c}_1-\mathbf{c}_2)dt\cong\mathbf{r}+\frac{D}{2}\mathbf{k}.$$

Being $\mathbf{k}=-\mathbf{j}_1$, then $d\mathbf{k}=d\mathbf{j}_1$ and Eq. (16) can be rewritten as

$$f^{(1)}(\mathbf{c}_1,\mathbf{r}_1,t)d\mathbf{c}_1\left(\frac{D}{2}\right)^2 dR\,d\mathbf{j}_1=f^{(1)}(\mathbf{c}_1,\mathbf{r}_1,t)d\mathbf{c}_1\left(\frac{D}{2}\right)^2 dR\,d\mathbf{k}. \tag{16'}$$

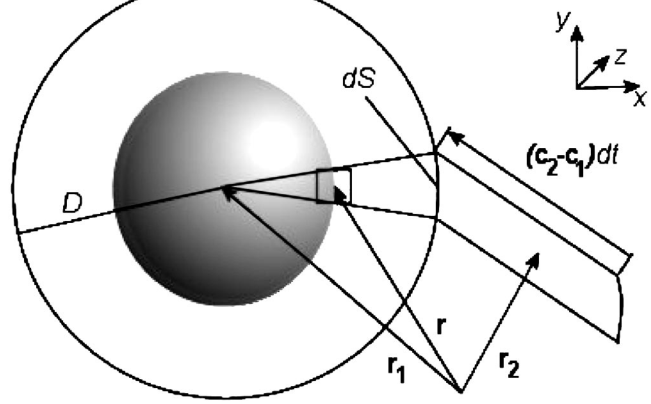

FIG. 5. Relative position of particles that are going to collide during $t\rightarrow t+dt$: the first is located in $\mathbf{r}_1$ while the second is in a point $\mathbf{r}_2$ before impact, such that the vector $\mathbf{k}$ linking the particle's centers at the impact passes through $\mathbf{r}$, $d\mathbf{r}$. The view is bidimensional (see Fig. 4).

The probability $\underline{P}_{\mathrm{coll}}$ in the unit of time and volume, in the hypothesis of independence of the two events, is given, accordingly to Eq. (3), by the product of the two probabilities corrected by the radial distribution function g_0,

$$\underline{P}_{\mathrm{coll}}=g_0(\mathbf{r},t)f^{(1)}(\mathbf{c}_1,\mathbf{r}_1,t)d\mathbf{c}_1 f^{(1)}(\mathbf{c}_2,\mathbf{r}_2,t)d\mathbf{c}_2\{\mathbf{g}\cdot\mathbf{k}\}D^2 d\mathbf{k}, \tag{17}$$

with $\mathbf{g}=\mathbf{c}_1-\mathbf{c}_2$.

The cases described by $\underline{P}_{\mathrm{coll}}$ in which a collision during t,dt could happen must satisfy the condition $(\mathbf{g}\cdot\mathbf{k})>0$, which means that only approaching particles are considered, and not departing ones, since these will not collide. Once $\underline{P}_{\mathrm{coll}}$ has been specified, it is easy to calculate an expression for the collisional source of a property $\chi(\psi)$, that is,

$$\chi(\psi)=\frac{1}{2}\int\int\int_{\mathbf{g}\cdot\mathbf{k}>0}[(\psi_1'+\psi_2')-(\psi_1+\psi_2)]\underline{P}_{\mathrm{coll}}, \tag{18}$$

where ψ' is the property after the collision, where the coefficient 1/2 has been introduced to avoid counting the same collision twice. Therefore,

$$\chi(\psi)=\frac{1}{2}\int\int\int_{\mathbf{g}\cdot\mathbf{k}>0}[(\psi_1'+\psi_2')-(\psi_1+\psi_2)]g_0(\mathbf{r},t)\times f^{(1)}(\mathbf{c}_1,\mathbf{r}_1,t)d\mathbf{c}_1 f^{(1)}(\mathbf{c}_2,\mathbf{r}_2,t)d\mathbf{c}_2\{\mathbf{g}\cdot\mathbf{k}\}D^2 d\mathbf{k}, \tag{19}$$

where the integrations are for all values of $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{k}$ such that $(\mathbf{g}\cdot\mathbf{k})$ is positive. The result in Eq. (19) is similar to the expressions that could be obtained by the classical method of Chapman and Cowling [42]. The difference is due to the fact that we calculated P_{coll} making $\mathbf{r}$ (the point of evaluation of the P_{coll}) identify with the point in which the collision happens, keeping it fixed while particles 1 and 2 move around it; in this way the function g_0 can be evaluated at $\mathbf{r}$ and the positions of the particles 1 and 2, namely, $\mathbf{r}_1$ and $\mathbf{r}_2$, are symmetrical with respect to $\mathbf{r}$. We will take advantage of these features in the following evaluation of $\chi(\psi)$ (Sec. VI). Otherwise, applying properly the expressions of Chapman and Cowling [42], the fixed point is $\mathbf{r}_1$, while the point in which the collision happens and $\mathbf{r}_2$ move around $\mathbf{r}_1$; therefore $f^{(1)}(\mathbf{c}_1,\mathbf{r}_1=\mathbf{r},t)$ and $f^{(1)}(\mathbf{c}_2,\mathbf{r}_2=\mathbf{r}_1+D\mathbf{k},t)$ are evaluated in the shifted points and g_0 should be evaluated at the point of collision $\mathbf{r}_1+(D/2)\mathbf{k}$.

VI. THE EVALUATION OF $\chi(\psi)$

A Taylor expansion around $\mathbf{r}$ truncated to the first-order term is used to express $f^{(1)}(\mathbf{c}_1,\mathbf{r}_1,t)f^{(1)}(\mathbf{c}_2,\mathbf{r}_2,t)$ in terms of $f^{(1)}(\mathbf{c}_1,\mathbf{r},t), f^{(1)}(\mathbf{c}_2,\mathbf{r},t)$ and their spatial derivatives [47],

$$f^{(1)}(\mathbf{c}_1,\mathbf{r}_1,t)\cong f^{(1)}(\mathbf{c}_1,\mathbf{r},t)-\frac{D}{2}k_i\frac{\partial f^{(1)}(\mathbf{c}_1,\mathbf{r},t)}{\partial r_i}\bigg|_{\mathbf{c}_1,t} \tag{20}$$

being $\mathbf{r}_1=\mathbf{r}+(D/2)\mathbf{j}_1=\mathbf{r}-(D/2)\mathbf{k}$.
In the same way (considering $\mathbf{r}_2=\mathbf{r}+\mathbf{k}D/2$)

$$f^{(1)}(\mathbf{c}_2,\mathbf{r}_2,t)\cong f^{(1)}(\mathbf{c}_2,\mathbf{r},t)+\frac{D}{2}k_i\frac{\partial f^{(1)}(\mathbf{c}_2,\mathbf{r},t)}{\partial r_i}\bigg|_{\mathbf{c}_2,t}. \tag{21}$$

Therefore,

$$\begin{aligned}f^{(1)}(\mathbf{c}_1,\mathbf{r}_1,t)f^{(1)}(\mathbf{c}_2,\mathbf{r}_2,t)&\cong f^{(1)}(\mathbf{c}_1,\mathbf{r},t)f^{(1)}(\mathbf{c}_2,\mathbf{r},t)\\&\quad+\frac{D}{2}k_i\frac{\partial f^{(1)}(\mathbf{c}_2,\mathbf{r},t)}{\partial r_i}\bigg|_{\mathbf{c}_2,t}f^{(1)}(\mathbf{c}_1,\mathbf{r},t)\\&\quad-\frac{D}{2}k_i\frac{\partial f^{(1)}(\mathbf{c}_1,\mathbf{r},t)}{\partial r_i}\bigg|_{\mathbf{c}_2,t}f^{(1)}(\mathbf{c}_2,\mathbf{r},t),\end{aligned} \tag{22}$$

if the second-order terms can be neglected.

Introducing the result of Eq. (22) in Eq. (19) and if a good approximation for $f^{(1)}$ is given by Eq. (5) truncated to the second-order terms (expressed in terms of actual velocities), the collisional source of a property $\chi(\psi)$ can be obtained as a sum of four terms,

$$\chi(\psi)=E(\psi)+F(\psi)+b_{ij}F_{ij}(\psi)+G(\psi), \tag{23}$$

where

$$E(\psi)=2g_0(\mathbf{r},t)\int\int\int\Delta(\psi)f_{01}f_{02}\left(\frac{D}{2}\right)^2(\mathbf{g}\cdot\mathbf{k})d\mathbf{k}\,d\mathbf{c}_1\,d\mathbf{c}_2, \tag{24}$$

$$\begin{aligned}F(\psi)=g_0(\mathbf{r},t)\int\int\int&\left\{f_1\frac{\partial f_2}{\partial r_m}\bigg|_{\mathbf{c}_2,t}-f_2\frac{\partial f_1}{\partial r_m}\bigg|_{\mathbf{c}_1,t}\right\}\Delta(\psi)k_m\\&\times2\left(\frac{D}{2}\right)^3(\mathbf{g}\cdot\mathbf{k})d\mathbf{k}\,d\mathbf{c}_1\,d\mathbf{c}_2,\end{aligned} \tag{24$'$}$$

$$\begin{aligned}F_{ij}(\psi)=2g_0(\mathbf{r},t)\int\int\int f_{01}f_{02}\Delta(\psi)\left(\frac{D}{2}\right)^2(\mathbf{g}\cdot\mathbf{k})\\\times\left\{\frac{[c_{2i}-u_i(\mathbf{r},t)][c_{2j}-u_j(\mathbf{r},t)]}{T}\right.\\\left.+\frac{[c_{1i}-u_i(\mathbf{r},t)][c_{1j}-u_j(\mathbf{r},t)]}{T}-2\delta_{ij}\right\}\\\times d\mathbf{k}\,d\mathbf{c}_1\,d\mathbf{c}_2,\end{aligned} \tag{24$''$}$$

$$\begin{aligned}G(\psi)=2g_0(\mathbf{r},t)b_{ab}b_{st}\int\int\int\Delta(\psi)f_{01}f_{02}\left(\frac{D}{2}\right)^2(\mathbf{g}\cdot\mathbf{k})\\\times\left\{\frac{[c_{2a}-u_a(\mathbf{r},t)][c_{2b}-u_b(\mathbf{r},t)]}{T}-\delta_{ab}\right\}\\\times\left\{\frac{[c_{1s}-u_s(\mathbf{r},t)][c_{1t}-u_t(\mathbf{r},t)]}{T}-\delta_{st}\right\}\\\times d\mathbf{k}\,d\mathbf{c}_1\,d\mathbf{c}_2,\end{aligned} \tag{24$'''$}$$

where we used the following abbreviations:

$$f_{02}=f_0^{(1)}(\mathbf{c}_2,\mathbf{r},t),$$

$$f_{01}=f_0^{(1)}(\mathbf{c}_1,\mathbf{r},t),$$

$$f_2=f^{(1)}(\mathbf{c}_2,\mathbf{r},t),$$

$$f_1=f^{(1)}(\mathbf{c}_1,\mathbf{r},t),$$

$$\Delta\psi=(\psi_1'+\psi_2')-(\psi_1+\psi_2).$$

Note that all the involved integrals have to be calculated for $\mathbf{g}\cdot\mathbf{k}>0$.

The integral term F can be written as the sum of other nine terms, since

$$\left\{f_1\frac{\partial f_2}{\partial r_m}\bigg|_{\mathbf{c}_2,t}-f_2\frac{\partial f_1}{\partial r_m}\bigg|_{\mathbf{c}_1,t}\right\}k_m\left(\frac{D}{2}\right)=a+b+c+d+e+f+g+h+i,$$

where

$$\begin{aligned}a=\left(\frac{D}{2}\right)k_m\frac{1}{T}f_{01}^{(1)}f_{02}^{(1)}\frac{\partial b_{st}}{\partial r_m}[(c-u)_{2s}(c-u)_{2t}\\-(c-u)_{1s}(c-u)_{1t}],\end{aligned}$$

$$\begin{aligned}b=\left(\frac{D}{2}\right)\frac{1}{T}f_{01}^{(1)}f_{02}^{(1)}\frac{\partial b_{st}}{\partial r_m}k_m\frac{b_{ij}}{T}\\\times[-(c-u)_{1s}(c-u)_{1t}(c-u)_{2i}(c-u)_{2j}\\+(c-u)_{1i}(c-u)_{1j}(c-u)_{2s}(c-u)_{2t}]\end{aligned}$$

$$\begin{aligned}c=f_{01}^{(1)}f_{02}^{(1)}\left(-\frac{1}{T}\right)\left[\frac{\partial\mathbf{u}}{\partial r_m}\cdot\mathbf{g}-\frac{1}{T}\frac{\partial T}{\partial r_m}\mathbf{u}\cdot\mathbf{g}\right.\\\left.+\frac{1}{T}\frac{\partial T}{\partial r_m}\mathbf{G}_0\cdot\mathbf{g}\right]k_m\left(\frac{D}{2}\right),\end{aligned}$$

$$d=f_{01}^{(1)}f_{02}^{(1)}\frac{b_{st}}{T}[(c-u)_{2s}(c-u)_{2t}+(c-u)_{1s}(c-u)_{it}]$$
$$\times k_m\left(\frac{D}{2}\right)\left(-\frac{1}{T}\right)\left[\frac{\partial \mathbf{u}}{\partial r_m}\cdot\mathbf{g}-\frac{1}{T}\frac{\partial T}{\partial r_m}\mathbf{u}\cdot\mathbf{g}+\frac{1}{T}\frac{\partial T}{\partial r_m}\mathbf{G}_0\cdot\mathbf{g}\right],$$

$$e=f_{01}^{(1)}\frac{b_{st}}{T}[(c-u)_{2s}(c-u)_{2t}]f_{02}^{(1)}\frac{b_{ab}}{T}[(c-u)_{1a}(c-u)_{1b}]$$
$$\times k_m\left(\frac{D}{2}\right)\left(-\frac{1}{T}\right)\left[\frac{\partial \mathbf{u}}{\partial r_m}\cdot\mathbf{g}-\frac{1}{T}\frac{\partial T}{\partial r_m}\mathbf{u}\cdot\mathbf{g}+\frac{1}{T}\frac{\partial T}{\partial r_m}\mathbf{G}_0\cdot\mathbf{g}\right]$$

$$f=f_{01}^{(1)}f_{02}^{(1)}\left(\frac{b_{ij}}{T}\right)\left\{b_{st}H_{st}(\mathbf{c}_1)\left[u_iu_j\left(-\frac{\partial\ln T}{\partial r_m}\right)+\frac{\partial u_iu_j}{\partial r_m}\right.\right.$$
$$-c_{2i}\left(\frac{\partial u_j}{\partial r_m}-u_j\frac{\partial \ln T}{\partial r_m}\right)-c_{2j}\left(\frac{\partial u_i}{\partial r_m}-u_i\frac{\partial \ln T}{\partial r_m}\right)-b_{st}H_{st}(\mathbf{c}_2)$$
$$\times\left[u_iu_j\left(-\frac{\partial \ln T}{\partial r_m}\right)+\frac{\partial u_iu_j}{\partial r_m}-c_{1i}\left(\frac{\partial u_j}{\partial r_m}-u_j\frac{\partial \ln T}{\partial r_m}\right)\right.$$
$$\left.\left.-c_{1j}\left(\frac{\partial u_i}{\partial r_m}-u_i\frac{\partial \ln T}{\partial r_m}\right)\right\}k_m\left(\frac{D}{2}\right),$$

$$g=f_{01}^{(1)}f_{02}^{(1)}\left(\frac{b_{ij}}{T}\right)\left\{\left(\frac{\partial u_j}{\partial r_m}-u_j\frac{\partial \ln T}{\partial r_m}\right)g_i\right.$$
$$\left.+g_j\left(\frac{\partial u_i}{\partial r_m}-u_i\frac{\partial \ln T}{\partial r_m}\right)\right\}k_m\left(\frac{D}{2}\right),$$

$$h=-f_{01}^{(1)}f_{02}^{(1)}\left(\frac{b_{ij}}{T}\right)\frac{\partial \ln T}{\partial r_m}\{c_{2i}c_{2j}-c_{1i}c_{1j}\}k_m\left(\frac{D}{2}\right),$$

$$i=-f_{01}^{(1)}f_{02}^{(1)}\left(\frac{b_{ij}}{T}\right)\frac{\partial \ln T}{\partial r_m}\{b_{st}H_{si}(\mathbf{c}_1)c_{2i}c_{2j}$$
$$-b_{st}H_{st}(\mathbf{c}_2)c_{1i}c_{1j}\}\mathbf{k}_m\left(\frac{D}{2}\right),$$

and $\mathbf{G}_0=(\mathbf{c}_1+\mathbf{c}_2)/2$. Collecting all these pieces we finally obtain F as the sum of nine integral terms,

$$F(\psi)=F_a(\psi)+F_b(\psi)+F_c(\psi)+F_d(\psi)$$
$$+F_e(\psi)+F_f(\psi)+F_g(\psi)+F_h(\psi)+F_i(\psi).$$

VII. THE CLOSURE EQUATIONS

A generic flow of granular material can be simulated using the balances of mass (12), of momentum (13), and of the second-order moments (14), (15). In these equations, the third-order moments (and their derivatives) also appear and, more generally, balance equations up to the Nth order always involve the $(N+1)$th-order moments (and their derivatives). These quantities have to be expressed as functions of the other variables (density, three components of the mean velocity, moments until the Nth order) in order to solve the system of equations. Frequently [63,47], the $(N+1)$th-order coefficients of the expansion (5) and their derivatives have been simply neglected because the $f^{(1)}$ has been approximated by the expansion (5) truncated to the Nth-order terms. However, we question the conclusion that, even in the case of negligible $(N+1)$th-order coefficients, their derivatives vanish as well. It is generally not true that a satisfactory approximation for $f^{(1)}$ turns out in a satisfactory approximation for the derivative of $f^{(1)}$. Therefore in the present section we develop new closure equations for the $(N+1)$th-order moments (and their derivatives) as functions of the lower-order terms. The solution followed by Grad [63] and by Jenkins and Richman [47] should be preferable if we are sure that the approximated $f^{(1)}$ fits sufficiently well the real distribution, so that the same truncation is adequate for the $f^{(1)}$ derivatives as well. Generally, however, this approach requires raising the order N at which the distribution function is truncated in Eq. (5), increasing the number of balance equations and variables involved and then the difficulties for calculating the collisional source term.

Let us recall that $M_{i_1i_2\cdots i_{N+1}}\equiv\langle C_{i_1}C_{i_2}\cdots C_{i_{N+1}}\rangle$ is equal to the generic $(N+1)$th-order moment of the fluctuating velocities divided by n; besides the generic $(N+1)$th-order moment of the fluctuating velocities

$$\int C_jC_{i_1}\cdots C_{i_N}f_C^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}$$

is equal to the diffusive flux along the j direction of the property $C_{i_1}\cdots C_{i_N}$ [42] and can be easily calculated once

$$\int C_jc_{i_1}\cdots c_{i_N}f_C^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C} \tag{25}$$

is known. As an example, the lower terms $(N=1,N=2)$ result in

$$\int C_jC_{i_1}f_C^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}=\int C_jc_{i_1}f_C^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}, \tag{26}$$

$$\begin{aligned}\int C_jC_{i_1}C_{i_2}f_C^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}\\ =\int C_jc_{i_1}c_{i_2}f_C^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}\\ -u_{i_1}\int C_jc_{i_2}f_C^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}\\ -u_{i_2}\int C_jc_{i_1}f_C^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}.\end{aligned} \tag{26$'$}$$

Again, the integral (25) is equal to the diffusive flux along the j direction of the property $c_{i_1}\cdots c_{i_N}$ [42]. In the following we will construct an approximate expression for such diffusive fluxes, based on some concepts of the "elementary kinetic theory,"

$$\begin{aligned}&(\text{diffusive flux of the property } c_{i_1}\cdots c_{i_N} \text{ on the generic } j \text{ direction})\\&=(\text{positive diffusive flux of particles on the } j \text{ direction})\\&\times(\text{difference of the mean value of the property } c_{i_1}\cdots c_{i_N}).\end{aligned} \tag{27}$$

The positive diffusive flux of particles on the j direction has been calculated using the expansion of Eq. (5) truncated to the second-order terms,

$$\int_{C_j\geq 0} C_j f_C^{(1)}(\mathbf{C},\mathbf{r},t)\,d\mathbf{C}=n\sqrt{\frac{T}{2\pi}}(1+b_{jj}). \tag{28}$$

We call ΔP the difference of the mean value of the property $c_{i_1}\cdots c_{i_N}$. To determine it, we consider an area element dA and we set in $\underline{\mathbf{r}}$ a reference frame $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ (Fig. 6); $\underline{\mathbf{r}}$ is the point where we want to calculate the fluxes. We focus on the particles that in the time interval $t\rightarrow t+dt$ go through the area element dA. These particles are those which at the time t are inside the volume $dA\,dt\,\mathbf{c}$, $\mathbf{c}$ being their velocity. The measure of the volume element $dA\,dt\,\mathbf{c}$ is $dA\,dt|\mathbf{c}|\cos\psi$, ψ being the angle between $\mathbf{j}$ and $\mathbf{c}$. ΔP is equal to the mean value of the property $c_{i_1}\cdots c_{i_N}$ conveyed through dA in the time interval dt by particles moving in the "positive direction," i.e., according to the unit vector $\mathbf{j}$, minus the same quantity for those moving in the "negative direction." ΔP will be estimated by $P^{j+}-P^{j-}$, where P^{j+} (and, respectively, P^{j-}) is the mean value of the property $c_{i_1}\cdots c_{i_N}$, over the values $c_j>0$, evaluated on the i-k plane (Fig. 6) where particles had, on the average, the last collision before passing through $\underline{\mathbf{r}}$.

The distance on the generic q direction between the mean position of the last collision of the particles that move in the "positive direction" with respect to $\underline{\mathbf{r}}$ is l_j^{j+} while l_j^{j-} is the distance along q in the "negative direction," therefore, P^{j+} has to be evaluated at l_j^{j+} (and, respectively, P^{j-} at l_j^{j-}). Then, neglecting external forces, the difference ΔP can be expressed as

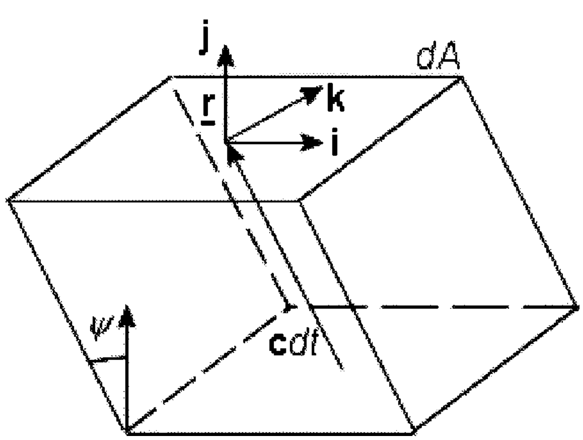

FIG. 6. Area element around $\underline{\mathbf{r}}$ and volume element $dA\,dt\,\mathbf{c}$ in which the particle that is flowing through dA with velocity $\mathbf{c}$ during dt is contained.

$$\Delta P=P^{j+}(l_j^{j+})-P^{j-}(l_j^{j-}). \tag{29}$$

Given the results (28) and (29), we can write the diffusive flux of the property $c_{i_1}\cdots c_{i_N}$ on the j direction, according to Eq. (27), as

$$n\sqrt{\frac{T}{2\pi}}(1+b_{jj})[P^{j+}(l_j^{j+})-P^{j-}(l_j^{j-})] \tag{27$'$}$$

or

$$\begin{aligned}&n\sqrt{\frac{T}{2\pi}}(1+b_{jj})[P(l_j^{j+})-P(l_j^{j-})]\\&\cong n\sqrt{\frac{T}{2\pi}}(1+b_{jj})\left[\left.\frac{\partial P}{\partial r_j}\right|_{\mathbf{r}=\underline{\mathbf{r}}}(l_j^{j+}-l_j^{j-})\right]\\&=\left[n\sqrt{\frac{T}{2\pi}}(1+b_{jj})\left.\frac{\partial\langle c_{i_1}\cdots c_{i_N}\rangle}{\partial r_j}\right|_{\mathbf{r}=\underline{\mathbf{r}}}(l_j^{j+}-l_j^{j-})\right],\end{aligned} \tag{27$''$}$$

if the average P^{j+} over the values $c_j>0$ (and, respectively P^{j-}) can be substituted by the mean P over all possible values of $\mathbf{c}$.

An expression for l_j^{j+} (and l_j^{j-}) as a function of density, three components of the mean velocity, and moments up to the Nth-order needs to be developed. Generally, l_q^{j+} can be calculated as the mean value of $(-c_q t_c)$, where c_q is the velocity at the point $\underline{\mathbf{r}}$ of particles crossing dA in dt and t_c is the interval between the time of the last collision and the time in which the particle arrives at the point $\underline{\mathbf{r}}$. This calculation can be approximated as the product of $\langle t_c^{j+}\rangle$ (the mean value of t_c over the particles that travel in the "positive direction") by $\langle c_q^{j+}\rangle^*$ (the mean value of c_q over all value of $c_j>0$). The first one, $\langle t_c^{j+}\rangle$, is overestimated by τ^{j+}, the collision interval (the mean time between two successive collisions) for particles in the "positive direction," so l_q^{j+} is calculated as

$$l_q^{j+}=-K_q^{j+}\tau^{j+}\langle c_q^{j+}\rangle^*,\quad 0\leq K_q^{j+}\leq 1. \tag{30}$$

This expression has been obtained neglecting the influence of the external forces on the particle moving towards the surface element dA, namely, considering that the particle travels to dA with the velocity c_q, that the particle has at the point $\underline{\mathbf{r}}$. Instead, in the case of a constant external force acting along the q direction, F_q, the velocity of the particle after the last collision (before crossing dA) is given by

$$c_q - \frac{F_q}{m} t_c,$$

therefore, the mean value of the velocity of the particle from the last collision to the surface element dA is

$$c_q - \frac{1}{2}\frac{F_q}{m} t_c,$$

because F_q is constant. In this hypothesis, Eq. (30) is replaced by

$$l_q^{j+} = -K_q^{j+}\tau^{j+}\left\{\langle c_q^{j+}\rangle^* - K_q^{j+}\frac{1}{2}\frac{F_q}{m}\tau^{j+}\right\}, \quad 0 \le K_q^{j+} \le 1. \tag{30$'$}$$

The collision interval τ in a specific $(\underline{\mathbf{r}},\underline{t})$ has been evaluated by a method similar to that provided by Chapman and Cowling [42], but using the expansion (5) truncated to the second-order terms to express the $f_C^{(1)}$ instead of the Maxwellian distribution function,

$$\tau = \frac{1}{D^2 n 4(\pi T)^{0.5}\left[1 - \frac{1}{60} b_{xy}^2\right]}, \tag{31}$$

the dependency upon $\underline{\mathbf{r}}$ and $\underline{t}$ is given through the quantities n (proportional to the density), T, b_{xy}.

Then, we can provide a first approximation of l_q^{j+} of Eq. (30$'$) with $\tau^{j+} = \tau(\underline{\mathbf{r}},\underline{t})$,

$$l_q^{j+} = -K_q^{j+}\tau(\underline{\mathbf{r}},\underline{t})\left\{\langle c_q^{j+}\rangle^* - K_q^{j+}\frac{1}{2}\frac{F_q}{m}\tau(\underline{\mathbf{r}},\underline{t})\right\},$$

$$0 \le K_q^{j+} \le 1. \tag{30$''$}$$

A better approximation can be obtained expressing τ^{j+} as the arithmetical mean of the values of τ at $(\underline{\mathbf{r}},\underline{t})$ and at the last collision, defined by $(\mathbf{l}^{j+},\tau^{j+})$,

$$\tau^{j+} = \frac{1}{2}[\tau(\underline{\mathbf{r}},\underline{t}) + \tau(\underline{\mathbf{r}} + \mathbf{l}^{j+}, \underline{t} - \tau^{j+})]$$

$$\cong \tau(\underline{\mathbf{r}},\underline{t}) + \frac{1}{2}\frac{\partial\tau}{\partial r_p}\bigg|_{\mathbf{r}=\underline{\mathbf{r}}} l_p^{j+} - \frac{1}{2}\frac{\partial\tau}{\partial t}\bigg|_{t=\underline{t}}\tau^{j+},$$

$$\tau^{j+} \cong \left[\tau(\underline{\mathbf{r}},\underline{t}) + \frac{1}{2}\frac{\partial\tau}{\partial r_p}\bigg|_{\mathbf{r}=\underline{\mathbf{r}}} l_p^{j+}\right]\frac{1}{\left(1 + \frac{1}{2}\frac{\partial\tau}{\partial t}\Big|_{t=\underline{t}}\right)}.$$

However, this introduces an additional dependency of τ^{j+} upon $\mathbf{l}^{j+}$, which is the quantity sought for. Consequently, in this second approximation, the terms l_q^{j+} result from the solution of the following algebraic equations:

$$l_q^{j+} = -K_q^{j+}\left[\tau(\underline{\mathbf{r}},\underline{t}) + \frac{1}{2}\frac{\partial\tau}{\partial r_p}\bigg|_{\mathbf{r}=\underline{\mathbf{r}}} l_p^{j+}\right]\frac{1}{\left(1 + \frac{1}{2}\frac{\partial\tau}{\partial t}\Big|_{t=\underline{t}}\right)}\left\{\langle c_q^{j+}\rangle^* - K_q^{j+}\frac{1}{2}\frac{F_q}{m}\left[\tau(\underline{\mathbf{r}},\underline{t}) + \frac{1}{2}\frac{\partial\tau}{\partial r_p}\bigg|_{\mathbf{r}=\underline{\mathbf{r}}} l_p^{j+}\right]\frac{1}{\left(1 + \frac{1}{2}\frac{\partial\tau}{\partial t}\Big|_{t=\underline{t}}\right)}\right\},$$

$$0 \le K_q^{j+} \le 1. \tag{30$'''$}$$

In the previous equations (30)–(30$'''$), we presented different approximations for l_q^{j+}; in the following calculations (Sec. VIII), we will use the expression of Eq. (30) with $\tau^{j+} = \tau(\underline{\mathbf{r}},\underline{t})$ because it is the easier to implement, though possibly less precise. The mean value $\langle c_q^{j+}\rangle^*$ of c_q over all values $c_j > 0$, for particles crossing dA in dt, is given by

$$\langle c_q^{j+}\rangle^* = \frac{\int_{c_j \ge 0} c_q f^{(1)}(\mathbf{c},\mathbf{r},t) d\mathbf{c}(dA\, dt|\mathbf{c}|\cos\psi)}{\int_{c_j \ge 0} f^{(1)}(\mathbf{c},\mathbf{r},t) d\mathbf{c}(dA\, dt|\mathbf{c}|\cos\psi)}$$

$$= \frac{\int_{c_j \ge 0} c_q f^{(1)}(\mathbf{c},\mathbf{r},t) c_j\, d\mathbf{c}}{\int_{c_j \ge 0}(\mathbf{c},\mathbf{r},t) c_j\, d\mathbf{c}}.$$

In the classical perspective this value can be calculated, with further approximation, as the average of $|\mathbf{c}|$ over all possible values of $\mathbf{c}$, multiplied by the average of the ratio between c_q and $|\mathbf{c}|$ $(=\cos\psi$ if $j=q)$ over the values of $\mathbf{c}$ such that $c_j > 0$. Besides, to calculate the mean value of $|\mathbf{c}|$ the Maxwell distribution function has been used and nonzero components of the mean velocity have been neglected. These reasonements bring to the classical expressions for l_q^{j+} [65], calculated by Eq. (30), in the case $j=q$,

$$l_j^{j+} = -\lambda\frac{2}{3},$$

where λ is the mean free path, $\tau^{j+} = \tau(\underline{\mathbf{r}},\underline{t})$ and K_j^{j+} is assumed equal to l.

Differently, we calculated more rigorously the average $\langle c_q^{j+}\rangle^*$, taking advantage of the expansion for $f_C^{(1)}$ of Eq. (5) truncated to the second-order terms (with $\mathbf{C} = \mathbf{c} - \mathbf{u}$) to express the $f^{(1)}$. Specifically, we obtained explicit solutions (besides the error function) for both the numerator and the denominator, as follows.

(1) For $j=q$.

$$\langle c_q^{j+}\rangle^* = \frac{\int_{c_j\geq 0} c_j^2 f^{(1)}(\mathbf{c},\mathbf{r},t)d\mathbf{c}}{\int_{c_j\geq 0} f^{(1)}(\mathbf{c},\mathbf{r},t)c_j d\mathbf{c}} \tag{32}$$

with

$$\int_{c_j\geq 0} c_j^2 f^{(1)}(\mathbf{c},\mathbf{r},t)d\mathbf{c} = \frac{n}{(2\pi T)^{3/2}} \sum_{n=0}^{4} \alpha_n \int_{-u_j/\sqrt{2T}}^{+\infty} Y^n e^{-Y^2} dY,$$

$$\alpha_0 = 2\pi T u_j^2 \sqrt{2T} - \frac{1}{T} b_{jj} u_j^2 \frac{\pi}{2}(T2)^{2.5},$$

$$\alpha_1 = 8\pi T^2 u_j - \frac{1}{T} b_{jj} u_j \pi (T2)^3,$$

$$\alpha_2 = 2\pi T (T2)^{3/2} - \frac{1}{T} b_{jj} \frac{\pi}{2}(T2)^{3.5} + \frac{1}{T} b_{jj} u_j^2 \pi (T2)^{2.5},$$

$$\alpha_3 = \frac{1}{T} b_{jj} 2 u_j \pi (T2)^3,$$

$$\alpha_4 = \frac{1}{T} b_{jj} \pi (T2)^{3.5},$$

and

$$\int_{c_j\geq 0} f^{(1)}(\mathbf{c},\mathbf{r},t)c_j d\mathbf{c} = \frac{n}{(2\pi T)^{3/2}} \sum_{n=0}^{4} \gamma_n \int_{-u_j/\sqrt{2T}}^{\infty} Y^n e^{-Y^2} dY,$$

$$\gamma_0 = 2\pi T u_j \sqrt{2T} - \frac{1}{T} b_{jj} u_j \frac{\pi}{2}(T2)^{2.5},$$

$$\gamma_1 = 4\pi T^2 - \frac{1}{T} b_{jj} \frac{\pi}{2}(T2)^3,$$

$$\gamma_2 = \frac{1}{T} b_{jj} u_j \pi (T2)^{2.5},$$

$$\gamma_3 = \frac{1}{T} b_{jj} \pi (T2)^3.$$

(2) For $j\neq q$.

$$\langle c_q^{j+}\rangle^* = \frac{\int_{c_j\geq 0} c_q f^{(1)}(\mathbf{c},\mathbf{r},t)c_j\, d\mathbf{c}}{\int_{c_j\geq 0} f^{(1)}(\mathbf{c},\mathbf{r},t)c_j\, d\mathbf{c}}, \tag{33}$$

with

$$\int_{c_j\geq 0} c_q f^{(1)}(\mathbf{c},\mathbf{r},t)c_j d\mathbf{c} = \frac{n}{(2\pi T)^{3/2}} \sum_{n=0}^{3} \delta_n \int_{-u_j/\sqrt{2T}}^{\infty} Y^n e^{-Y^2} dY,$$

$$\delta_0 = 2\pi T u_j u_q \sqrt{2T} - \frac{1}{T} b_{jj} u_j u_q \frac{\pi}{2}(T2)^{2.5},$$

$$\delta_1 = 4\pi T^2 u_q - \frac{1}{T} b_{jj} u_q \frac{\pi}{2}(T2)^3 + \frac{1}{T} b_{qj} u_j \frac{\pi}{2}(T2)^3,$$

$$\delta_2 = \frac{1}{T} b_{qj} \frac{\pi}{2}(T2)^{3.5} + \frac{1}{T} b_{jj} u_q u_j \pi (T2)^{2.5},$$

$$\delta_3 = \frac{1}{T} b_{jj} u_q \pi (T2)^3,$$

and, for the denominator, the same expression obtained for $j=q$ holds. Note that all the integrals involved $\int_{-u_j/\sqrt{2T}}^{+\infty} Y^n e^{-Y^2} dY$ can be evaluated analytically, if $u_j=0$, or through the standard error function. Some details are reported in the Appendix.

Summarizing, knowledge of $\langle c_q^{j+}\rangle^*$ and τ^{j+} allows to determine l_q^{j+} (and, similarly, l_q^{j-}) with Eqs. (30)–(30′′) and then, through Eqs. (27), we can evaluate the diffusive fluxes or the $(N+1)$th-order moments as functions of the density, three components of the mean velocity, and moments until the Nth order.

VIII. THE TRANSLATIONAL REGIME IN A CHUTE FLOW

Jenkins and Richman [47] used the Grad's method as a basic scheme to study granular dynamics. They tried to obtain a general solution, valid for any kind of motion, for the constitutive equations of the coefficients of the series expansion (5), but they said that the procedure was valid in the limit $e\to 1$ (e is the restitution coefficient).

Here, we follow a different approach searching a solution for a specific configuration, but without restrictions on e. The procedure used can be applied in a similar way to any other kind of flow, but is expected to result in different solutions, instead of a single, general one.

We concentrate on the fully developed, gravity driven, stationary granular flow in an inclined chute. The coordinate system is sketched in Fig. 7. It is supposed that the influence of the confining walls is negligible, so that there are no variations of the quantities along z (z-independent flux). We address to the translational regime, i.e., we will neglect the collisional terms of flux of a property Θ_i with respect to the translational ones. In order to extend the model to simulate also the collisional regime, neglected fluxes must be considered. On the contrary, the collisional source term χ cannot be anticipated to be negligible with respect to the derivative of the translational flux terms or to the product of the translational flux terms by the mean velocity gradient [Eq. (11)].

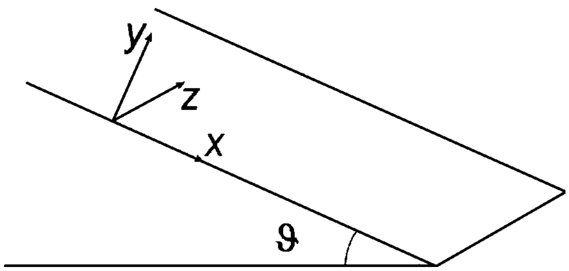

FIG. 7. Sketch of the chute geometry with reference frame.

Accordingly, we keep considering χ also with the dilute flow.

For the granular type of flow considered here, the balance of mass (12) and momentum along the z coordinate are identically satisfied, while the momentum balances (13) along the x and y coordinates reduce to

$$\rho g_x = \frac{\partial \rho T b_{xy}}{\partial y}, \tag{34}$$

$$\rho g_y = \frac{\partial \rho T(1+2b_{yy})}{\partial y}. \tag{35}$$

The balance of the hydrostatic part of the second-order moments [Eq. (14)] becomes

$$\frac{1}{2}\sum_{j=x,y,z}\chi_{jj} = \frac{\partial u_x}{\partial y}\rho T b_{xy} + \frac{1}{2}\sum_{j=x,y,z}\frac{\partial}{\partial y}(\rho M_{yjj}), \tag{36}$$

three equations among the Eqs. (15) reduce to

$$\rho T \frac{\partial u_x}{\partial y}(1+2b_{yy}) = \chi_{xy} - \frac{\partial}{\partial y}(\rho M_{yxy}), \tag{37}$$

$$-\frac{2}{3}\rho T\frac{\partial u_x}{\partial y}b_{xy} + \frac{\partial\left(\rho M_{yyy} - \frac{1}{3}\Sigma_{j=x,y,z}\rho M_{yjj}\right)}{\partial y} = \chi_{yy} - \frac{1}{3}\sum_{j=x,y,z}\chi_{jj}, \tag{38}$$

$$-\frac{2}{3}\rho T\frac{\partial u_x}{\partial y}b_{xy} + \frac{\partial\left(\rho M_{zzy} - \frac{1}{3}\Sigma_{j=x,y,z}\rho M_{yjj}\right)}{\partial y} = \chi_{zz} - \frac{1}{3}\sum_{j=x,y,z}\chi_{jj}. \tag{39}$$

Equations (37), (38), (39) are, respectively, the balances of (C_xC_y), $(C_yC_y - C^2/3)$, $(C_zC_z - C^2/3)$. Balance of $(C_xC_x - C^2/3)$ is dependent on the balances (38), (39).

To obtain the previous results, it has been considered that in the case of a fully developed, stationary, z-independent chute flow, b_{yz} and b_{xz} are equal to zero, because the diffusive translational momentum fluxes along the z coordinate ρM_{yz} and ρM_{xz}, related to b_{yz} and b_{xz} by Eq. (8), are also null; moreover, u_y and u_z are zero. Once the terms of collisional source χ_{ij}, the M_{yij} and the M_{yxy}, are expressed as functions of variables ρ, u_x, T, b_{xy}, two independent coefficients among the b_{ii}, and their derivatives, Eqs. (34)–(39) become a system of ordinary differential equations (ODEs) with respect to the independent variable y, to be solved in the six dependent variables ρ, u_x, T, b_{xy}, and two independents coefficients among the b_{ii}. To solve the ODEs system, some pieces of information must be additionally given, namely, χ_{ij}, M_{yjj}, M_{yxy}, and appropriate boundary conditions.

A. The evaluation of χ_{ij}

The collisional source term χ_{ij} has to be expressed as a function of ρ, u_x, T, b_{xy}, and two independents coefficients among the b_{ii}. To explicitly determine these dependencies, we used Eqs. (23) and (24)–(24′′′), which have been derived introducing Eqs. (22) and (5) truncated to the second-order terms in Eq. (19). A detailed investigation should be useful to understand how much this approximation fits the real value of the product $f^{(1)}(\mathbf{c}_1,\mathbf{r}_1,t)f^{(1)}(\mathbf{c}_1,\mathbf{r}_2,t)$.

Here, we just observe that Drake [67] and Azanza, Chevoir, and Moucheront [66] showed experimentally that for the kind of flow considered here, $f^{(1)}$ is an anisotropic quasi-Maxwellian. Accordingly, the $f^{(1)}$ expansion of Eq. (5) truncated to the second-order terms is a good approximation to calculate χ_{ij}. Despite the fact that the experiments of both were performed in a bidimensional (only one layer of particles in the z direction) channel, the qualitative features of the flow is expected to be the same for three-dimensional channels.

An even more precise expression would be given by Eq. (5) truncated to the third-order terms. The only terms of the third order to be considered in this configuration of flow would be those with the coefficients b_{yjj}. The others can be neglected because the generic b_{ijk} is proportional to M_{ijk} and the terms M_{xyz} and M_{zjj} can be taken equal to zero because of the symmetry of $f^{(1)}$ with respect to C_z while the terms M_{xjj} are proportional to the diffusive fluxes of the property M_{jj} and these can be evaluated as equal to zero (this will be shown in Sec. VIII B).

However, the approximation of the expansion truncated to the second-order terms brings to quite complex expressions, and the truncation to the third-order terms is even worse, increasing the chances of computing errors. Consequently, we remained with the second-order truncation of the $f^{(1)}$ for the following developments.

Therefore, the expressions (23) and (24)–(24′′′) have been used to calculate the general expressions for χ_{ij} and the results, in the case of flow considered here $[u_x = u_x(y),\ T = T(y),\ b_{yz} = b_{yz} = 0]$, are reported in the Appendix. The calculations have been carried out taking into account the difference in the mean velocity between two colliding particles and all the derivatives of the mean quantities that appear in $f^{(1)}$.

B. The evaluation of M_{yii} and M_{yxy}

In the following, we will consider the averages over the values $c_j > 0$ (Sec. VII) of some particle properties $(c_x, c_xc_x, c_yc_y, c_zc_z)$. For the kind of flow discussed here, such averages can be approximated by the means over all possible values of $\mathbf{c}$. Therefore, from Eqs. (26), (26′), and (27″), we obtain

$$M_{yii}=\frac{1}{n}\times(\text{diffusive flux of the property } C_iC_i \text{ on the } y \text{ direction})$$
$$=\int C_yC_iC_if_C^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}=\int C_yc_ic_if_C^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}-u_i\int C_yc_if_C^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}-u_i\int C_yc_if_C^{(1)}(\mathbf{C},\mathbf{r},t)d\mathbf{C}$$
$$=\sqrt{\frac{T}{2\pi}}(1+b_{yy})\left[\frac{\partial\langle c_ic_t\rangle}{\partial y}(l_y^{y+}-l_y^{y-})\right]-2u_i\sqrt{\frac{T}{2\pi}}(1+b_{yy})\left[\frac{\partial\langle c_i\rangle}{\partial y}(l_y^{y+}-l_y^{y-})\right]$$
$$=\sqrt{\frac{T}{2\pi}}(1+b_{yy})\left[\frac{\partial\langle C_iC_i\rangle}{\partial y}(l_y^{y+}-l_y^{y-})\right],$$

from Eqs. (30) and (32), we calculate

$$l_y^{y+}-l_y^{y-}=-\left(\frac{K_y^{y+}+K_y^{y-}}{2}\right)\tau\sqrt{2T\pi}\frac{(1+2b_{yy})}{(1+b_{yy})}$$

being

$$\langle c_y^{y+}\rangle^*=\frac{\int_{c_y\geq 0}c_y^2f^{(1)}(\mathbf{c},\mathbf{r},t)d\mathbf{c}}{\int_{c_y\geq 0}c_yf^{(1)}(\mathbf{c},\mathbf{r},t)d\mathbf{c}}=\frac{\sqrt{2T\pi}}{2}\frac{(1+2b_{yy})}{(1+b_{yy})}$$
$$=-\langle c_y^{y-}\rangle^*$$

and $\tau^{j+}=\tau^{j-}=\tau$.

Therefore,

$$M_{yii}=-\sqrt{\frac{T}{2\pi}}(1+2b_{yy})\left[\frac{\partial\langle C_iC_i\rangle}{\partial y}\tau\sqrt{2T\pi}\left(\frac{K_y^{y+}+K_y^{y-}}{2}\right)\right].$$

In the case $K_y^{y+}=K_y^{y-}=1$, it simplifies to

$$M_{yii}=-T(1+2b_{yy})\frac{\partial\langle C_iC_i\rangle}{\partial y}\tau.$$

Besides, from Eqs. (26), (26′), and (27″), we can develop the expression for M_{yxy},

$$M_{yxy}=M_{xyy}=\frac{1}{n}\times(\text{diffusive flux of the property } C_yC_y \text{ on the } x \text{ direction})$$
$$=\sqrt{\frac{T}{2\pi}}(1+b_{xx})\left[\frac{\partial\langle C_yC_y\rangle}{\partial x}(l_x^{x+}-l_x^{x-})\right]=0.$$

C. Boundary conditions and the comparison with the experimental data

The equation system (34)–(39), once the expressions for the collisional source χ_{ij}, M_{yjj}, and M_{yxy} previously calculated have been introduced, is a system of ordinary differential equations in the unknown functions $\rho(y)$, $u_x(y)$, $T(y)$, $b_{xy}(y)$, and two independents coefficients among the $b_{ii}(y)$. In these equations, the first derivatives of ρ, u_x, b_{xy} and the first and second derivatives of T and the two coefficients b_{ii} are involved. Therefore, we have to specify the boundary conditions for all the unknown functions and for the first derivatives of T and of two coefficients b_{ii}.

Since the model applies to the translational regime, which is typical of the uppermost layer of a granular flow, it is not so easy to provide "natural" boundary conditions for the unknown functions or for some of their first derivatives. One possibility is taking values from experimental measurements. Particularly, we used two sets of experimental data, i.e., those of Azanza, Chevoir, and Moucheront [66] and those of Drake [67]; the reason of this choice is that they present the

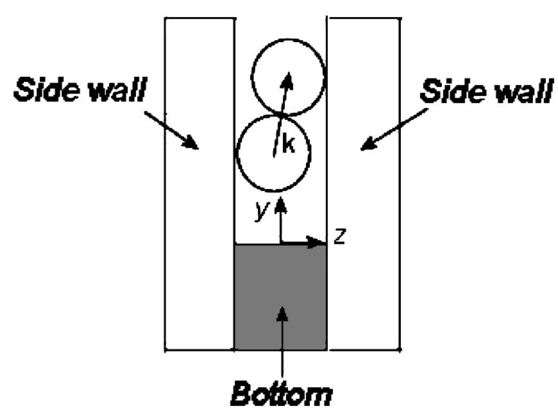

FIG. 8. Sketch of the bidimensional configuration used in Refs. [66] and [67] in their chutes; section at constant x.

TABLE I. Boundary conditions for the chute flow. Values used in the comparison with the data of Ref. [66] are reported in the third column, while in the fourth column data for the comparison with Ref. [67] are listed.

		Boundary conditions	
	Units	Comparison with data of Ref. [66]	Comparison with data of Ref. [67]
Solid volumetric fraction (ν)		0.027	0.034
Granular temperature (T)	m^2/s^2	0.145	0.265
X component of the mean velocity (u_x)	m/s	1.432	3.210
b_{yy}		−0.026	−0.058
b_{xx}		0.052	0.116
b_{xy}		−0.337	−0.525
$\frac{\partial T}{\partial y}$	m/s^2	0	0
$\frac{\partial b_{xx}}{\partial y}$	l/m	0	0
$\frac{\partial b_{yy}}{\partial y}$	l/m	0	0

more detailed series of measurements on the chute flow, to the best of our knowledge.

However, a problem is represented by the fact that both used chutes with sidewalls so close to allow for only one layer of particles (bidimensional channels), while our model is for three-dimensional channels, so the comparison between the experimental measures and theoretical results can be only approximate. Particularly, one of the differences is that the geometry chosen by Refs. [66], [67] constraints the particles to collide among each other only in a limited range of relative directions. In fact if **k**, the unit vector connecting the centres of two colliding particles (Fig. 8), is written in spherical coordinates, θ is the angle between **k** and y, and φ is the angle between the projection of **k** on the x-z plane and x. Because of the constraints given by the sidewalls in the experiments φ approaches zero, while in a three-dimensional geometry the particles can collide each other for any value of φ. Maybe this lower possibility of collision makes the flow in the bidimensional configuration faster (greater gradients of the mean velocity) than the flow in the three-dimensional geometry. Besides, other differences are the imperfect smoothness of the particles experimentally used and the influence of the walls, which tends to slow down the flow.

The assumed boundary conditions are listed in Table I. The basic criterion takes one experimental value at a given bed depth, from either Azanza, Chevoir, and Moucheront [66] (results with the chute angle equal to 23°) or Drake [67] (dilute flow). In both cases, values have been taken at the boundary, $y=y_b$, defined as the depth where the bidimensional solid fraction (surface fraction) is around 0.1, so that we are sure to fall within the translational regime [66]: in the case of Azanza, Chevoir, and Moucheront [66] the chosen y_b corresponds to $(y_b/D)=10.5$, in the case of Drake [67] it corresponds to $(y_b/D)=12$. The values of the quantities used for the boundary conditions are the values measured at $y=y_b$ or an interpolation of the neighboring experimental data.

However, further discussion is needed about the boundary conditions for the solid volumetric fraction, the two coefficients b_{ii}, and the coefficient b_{xy}. For the solid fraction, experimental data are expressed as bidimensional solid fraction ν_{2D} (particles surface on total surface) and they have been converted in the tridimensional volumetric solid fraction ν; we made the conversion through the approximate expression of Campbell and Brennen [72],

$$\nu=\nu_{2D}^{3/2}\frac{4}{3\sqrt{\pi}}.$$

Less obvious is the identification of appropriate boundary conditions for the two independent coefficients among the b_{ii}. Because of the bidimensional geometry, the experimental data for M_{zz} turns out to be zero [and, according to Eq. (8), the corresponding b_{zz} would be always $-1/2$]. This is equivalent to the assumption of negligible velocity fluctuations in the z direction, a condition definitely not verified in a three-dimensional configuration.

Therefore, we cannot use values obtained from the experimental measures of M_{ii} as boundary conditions for the b_{ii}. So we have been forced to introduce approximate values as boundary conditions for the b_{ii}, from an approximate solution of Eqs. (34)–(39). In the hypothesis that only the integrals E, F_a, F_b, F_c, F_{ij} are taken into account in the evaluation of χ_{ij}, Eqs. (38), (39) become

$$b_{yy}=-\frac{5(e-1)}{6(e-3)}, \tag{40}$$

$$b_{xx}=-2b_{yy}. \tag{41}$$

This result is obtained introducing in Eqs. (38), (39) the values of Table I for the other functions, particularly the derivatives of T and of the two coefficients b_{ii}. Also the collision interval τ has to be evaluated to derive Eqs. (40) and (41): the small values of b_{xy} (see point D) allow to simplify τ [Eq. (31)] by the following:

$$\tau=\frac{1}{D^2 n 4(\pi T)^{0.5}[1-\frac{1}{60}b_{xy}^2]}\cong\frac{1}{D^2 n 4(\pi T)^{0.5}}.$$

The coefficient b_{xy} is proportional to M_{xy} [Eq. (8)]. To our knowledge, no experimental data of such a quantity can be found in literature, so that some estimate of a boundary value for b_{xy} must be obtained theoretically. An approximate determination of the boundary condition for b_{xy} can be constructed from Eqs. (34) and (35), considering T constant with respect to y, as it is shown by the experimental data [66,67]. With this approximation, we obtained

$$\nu\{\tan\vartheta[1+2b_{yy}]+b_{yy}\}=\text{const}, \tag{42}$$

which provides an equation relating b_{xy} to b_{yy} and ν through an unknown constant.

The constant can be evaluated using the experimental values at the top of the flowing material where the gradient of the mean velocity goes to zero. Since, according to the "elementary kinetic theory" [Eq. (27″)], b_{xy} vanishes when the gradient of the mean velocity approaches zero, we can evaluate the constant of Eq. (42) at the surface of the flowing material,

$$\text{const}=\{\nu\tan\vartheta[1+2b_{yy}]\}_{\text{surface}}.$$

Since Eq. (42) holds throughout the whole flowing layer, we can use it to provide a value of b_{xy} at the bottom of the translational regime,

$$b_{xy}(y_b)=-\tan\vartheta[1+2b_{yy}(y_b)]+\frac{(\text{const})}{\nu(y_b)}, \tag{43}$$

where b_{yy} at y_b and at the surface was calculated as described above [Eq. (40)]. From the experimental values, we can calculate two distinct values of the constant,

$$\frac{(\text{const})_D}{\nu(y_b)}\approx 5\times 10^{-2},$$

in the case of Drake [67].

$$\text{const}_A/\nu(y_b)\approx 2\times 10^{-2},$$

in the case of Azanza, Chevoir, and Moucheront [66].

When we solve the system of ordinary differential equations (34)–(39) using for b_{xy} at y_b the approximate value calculated by Eq. (43), we obtain a value of b_{xy} at the top of the flowing layer different from zero, in contrast with the argument above. So we search in the surrounding of the value given by the Eq. (43) a new boundary condition for b_{xy} such that the solution of the system of ordinary differential Eqs. (34)–(39) gives b_{xy} vanishing at the top of the flowing layer. The relative difference between this value and the estimate with the previous Eq. (43) has been always less than 25%.

TABLE II. Values of the parameters and of the physical properties in the experimental studies as reported by the authors. The third column refers to the data of Ref. [66] with the chute angle equal to 23°, while the fourth column refers to those of Ref. [67] in the case of the dilute flow.

		Values of parameters	
	Units	Ref. [66]	Ref. [67]
Restitution coefficient		0.95 ± 0.03	0.84 ± 0.01
Chute angle	rad	0.401	0.746
Particle diameter	mm	3	6
Particle density	kg/(m^3)	7800	1319

D. Results

To carry out the simulations, χ_{ij} must be evaluated through the integrals in Eqs. (23) and (24)–(24‴), therefore an expression for g_0 [Eq. (3)] is required. We used the Verlet and Levesque formula [73],

$$g_0=\frac{(16-7\nu_{2D})}{16(1-\nu_{2D})^2}, \tag{44}$$

however, the correction introduced by the g_0 is small in the translational regime. The system of ordinary differential Eqs. (34)–(39) have been numerically integrated with the boundary conditions specified in Sec. VIII C. The values of the physical and geometrical properties and parameters used in the simulations are reported in the Table II. The results, in terms of profiles along y, have been plotted and, when available, compared with the experimental measurements.

The variation of the solid volumetric fraction measured by Azanza, Chevoir, and Moucheront [66] and Drake [67] is reported in Fig. 9, together with the corresponding prediction of our model. The agreement is quite good considering the absence of adaptive parameters in the model and the use of some approximation. Particularly, the exponential increase of the void ratio towards the surface is correctly reproduced. Note that only the uppermost layer is reported, where the regime can be assumed to be purely transnational. The integration of Eqs. (34)–(39) is done from y_b onwards, this being the depth from which clear transnational regime is observed [66]. It is also evident that the model overestimates the experimental solid fraction.

Mean velocity profiles are compared in Fig. 10. The measured parabolic profile is nicely reproduced by the model in both cases, though the measurements of Azanza, Chevoir, and Moucheront [66] are fairly noisy close to the surface of the flowing bed. Theoretical mean velocities are always smaller than measured. Note that, as far as the profiles of the solid volumetric fraction and the mean velocity are concerned, only their values at the boundary y_b are fixed (Table I), while their derivatives are always (also at the boundary) calculated by the model and not imposed. The granular tem-

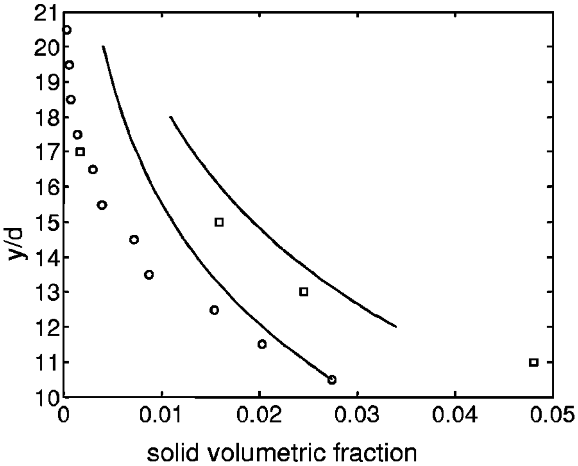

FIG. 9. Solid volumetric fraction profile in the translational layer of a granular flow down an inclined chute: theoretical results vs experimental data of Ref. [66] (circles) and Ref. [67] (squares).

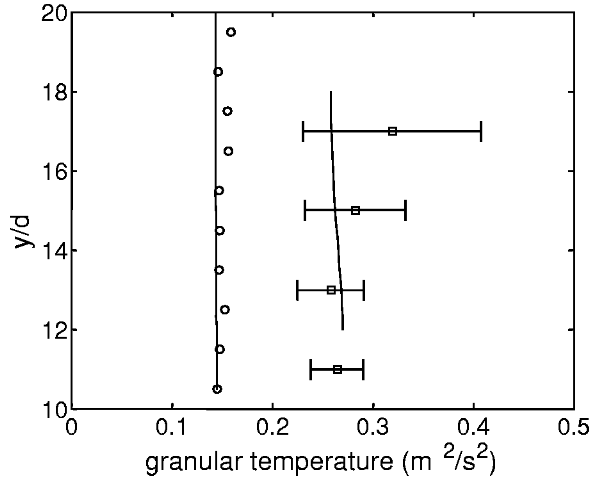

FIG. 11. Granular temperature profile in the translational layer of a granular flow down an inclined chute: theoretical results vs experimental data of Ref. [66] (circles) and of Ref. [67] (squares). The bars indicate the experimental errors in Ref. [67].

perature calculated by our model is compared with the measured ones in Fig. 11. Interestingly, the model predicts an almost constant value throughout the whole transnational layer, even thought T is solved as a function of y. Since also the measurements show a constant behavior and we used one experimental datum as boundary condition (for T and its derivative) the agreement is quite good. Some discrepancy can be seen with respect to the average results of Drake [67] in the higher region of the considered layer but in any case the theoretical results fall inside the experimental uncertainty reported in Ref. [67].

In Fig. 12 the theoretical profile for M_{yy}/M_{xx} is presented. Results of the simulation, corresponding to the experiments of Azanza, Chevoir, and Moucheront [66], are labeled by A in Fig. 12 and show an almost constant trend, as in the experimental work [66]. The values predicted by the model, around 0.83–0.86, are different with respect to the measurements, that range between 0.55 and 0.6. The reason of this difference has to be sought in the observations made in Sec. VIII C. There, we discussed some discrepancies between the system simulated by the theoretical model, i.e., the tridimensional chute flow with negligible walls influence (z-independent flux) and the bidimensional (only one layer of particles) experimental chute [66,67]. Particularly, in the experimental configuration $M_{zz}=0$, while in the tridimensional chute flow M_{zz} is clearly different from zero. So, for the same value of the particle internal energy [one measure of which is $(3/2)Tm$], i.e., for the same value of the granular temperature T, the way in which the total internal energy splits into its components (M_{xx},M_{yy},M_{zz}) in the bidimensional experiments and in the tridimensional model are expected to be similar only in a qualitative manner. According to this, we purposely rejected (Sec. VIII C) the experimental boundary conditions for b_{yy} and for b_{xx} (i.e., for M_{yy} and M_{xx}) as measured in a bidimensional configuration. Also Drake [67] reports for the ratio M_{yy}/M_{xx} a nearly constant value comprised between 0.39 and 0.51, while our model

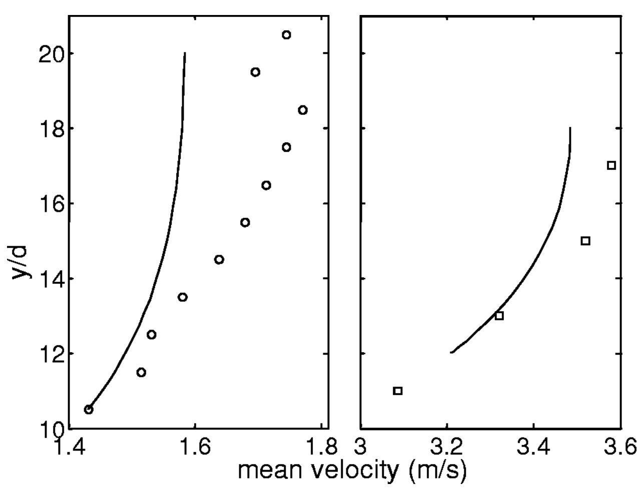

FIG. 10. X component of the mean velocity profile in the translational layer of a granular flow down an inclined chute: theoretical results vs experimental data of Ref. [66] at the left and those of Ref. [67] at the right.

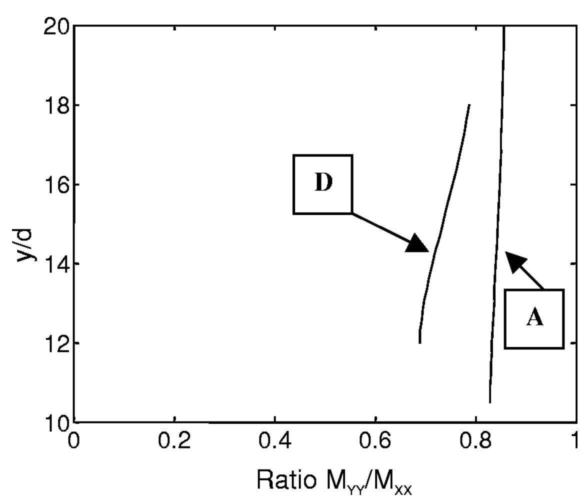

FIG. 12. M_{yy}/M_{xx} profile in the translational layer of a granular flow down an inclined chute: model predictions of the experiments of Ref. [67] (line D) and of Ref. [66] (line A).

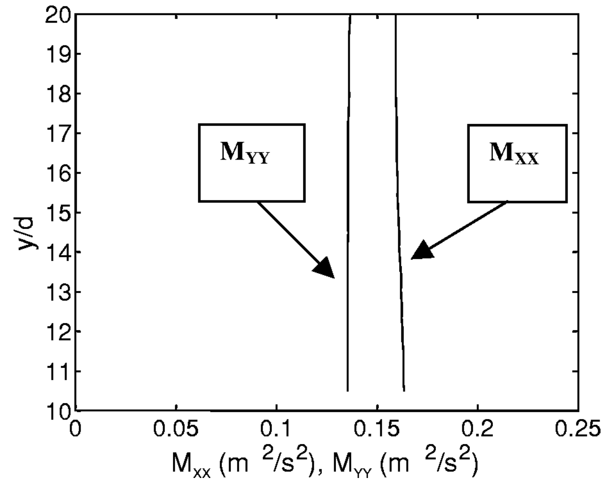

FIG. 13. M_{xx} and M_{yy} profiles in the translational layer of a granular flow down an inclined chute: model predictions of the experiments of Ref. [66].

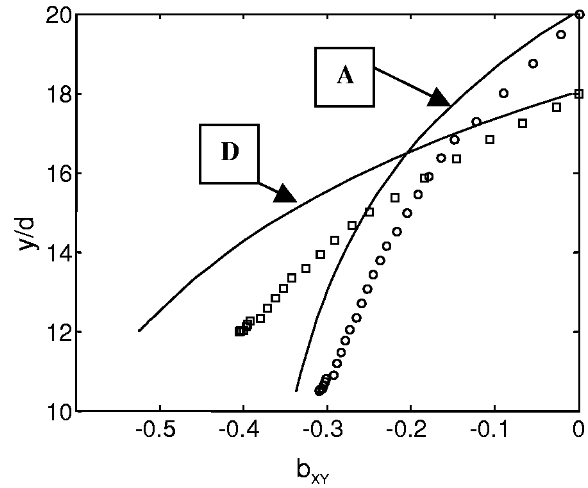

FIG. 14. Profile of the coefficient b_{xy} in the translational layer of a granular flow down an inclined chute. Referring to the experiments of Ref. [66], line A are the model predictions and circles are the predictions of Eq. (45); referring to the experiments of Ref. [67], line D are the model predictions and squares are the predictions of Eq. (45).

predicts values between 0.69 and 0.79, as shown by line D in Fig. 12.

In Fig. 13 we reported the theoretical profiles for both M_{yy} and M_{xx} obtained using parameters and boundary conditions corresponding to the experiments of Azanza, Chevoir, and Moucheront [66]. Like in the experimental work, both show a constant behavior. Moreover, M_{yy} (and also M_{zz}) is always smaller than T, while M_{xx} is always greater, i.e., the degree of the fluctuations along the x coordinate is greater than the degree of the fluctuations along any other direction. In other words, the translational pressures in the three directions predicted by the model are not equal and the granular matter is not in an hydrostatic state. Similar results have been obtained in the comparison with Ref. [67].

Finally, the theoretical profiles of the coefficient b_{xy}, proportional to the diffusive momentum flux ρM_{xy} according to Eq. (8), are shown in Fig. 14; unfortunately there are no experimental determinations available to compare with. In Fig. 14 line D refers to the simulation of the experiments of Drake [67], line A to those of Ref. [66]. In both cases, b_{xy} is negative, i.e., ρM_{xy} is correctly opposite to the y direction. Besides, the absolute value of b_{xy} decreases monotonically from the boundary condition fixed at y_b (Table I) down to zero at the free surface. The behavior of b_{xy} can be compared with the evaluation of the coefficient b_{xy} obtained by means of the "elementary kinetic theory," that we name b^*_{xy}. In fact, calculating the diffusive flux along y of the x momentum (mc_x) by Eqs. (27′)–(30) and dividing it by ρ and by T [Eq. (8)], one obtains,

$$b^*_{xy} = -\frac{\partial u_x}{\partial y}\tau(1+2b_{yy}), \tag{45}$$

with $\tau^{j+}=\tau^{j-}=\tau$ and $K_y^{y+}=K_y^{y-}=1$. The profiles of b^*_{xy} can be calculated introducing the simulation results of the derivative of u_x, ν, T, b_{xy}, and b_{yy} in Eq. (45). They are also reported in Fig. 14 where circles refer to the simulations of the experiments of Ref. [66], squares to those of Ref. [67]. In any case there is a good agreement between b_{xy} and the estimate b^*_{xy}, once more validating the detailed model developed. In this work the model has been compared with experimental data from Refs. [66], [67], having dissimilar values of the operating conditions (chute angle) and physical properties (particle diameter and density, restitution coefficient); different values of the parameters should be considered as well, above all lower restitution coefficients. Also the approximation used for the $f^{(1)}$ should be tested for highly dissipative particles. Besides, the consistency of the model should be verified with experimental measures free (as far as possible) from the limitations of the bidimensional geometry.

IX. CONCLUSIONS

The present work is devoted to modeling the rapid granular flows of smooth, identical spheres by means of the kinetic approach (granular temperature). In this field, while the basic concepts of the theory have been definitely explained, further efforts are needed in two directions.

The first is a departure from hypotheses valid only for ideal situations. In this sense, after having approximated the particle velocity distribution function $f^{(1)}$ by an expansion in Hermite polynomials around the Maxwellian truncated to the second order, we developed the collisional source term completely (not only in the integrals corresponding to the nearly elastic limit); moreover, the variation of the mean velocity between the two colliding particles has been taken into account. Besides, together with the mass and momentum balances, all the second-order moment balances have been considered [59]: these equations require the introduction of closure equations for the third-order moments (and their derivatives). Closure equations have been developed for the $(N+1)$th-order moments (and their derivatives) to be used if the Nth-order moment balances are considered, by a generalization of the "elementary kinetic theory" and specifically with a $f^{(1)}$ described by the Hermite expansion and with nonzero components of the mean velocity.

The second direction is to test the theoretical profiles with experimental data. We applied the model to the translational flow on an inclined chute and the simulations yielded the profiles of the solid volumetric fraction, the mean velocity, and all the second-order moments, expressed as granular temperature and coefficients of the approximated $f^{(1)}$.

The experimental data of Azanza, Chevoir, and Moucheront [66] and of Drake [67] have been considered, although they were obtained with a flow of just one layer of particles constrained by the walls (bidimensional geometry), so the comparison is only approximate. All the qualitative features of the flow are represented by the model: the decreasing (exponential) profile of the solid volumetric fraction, the parabolic shape of the mean velocity, the constancy of the granular temperature and of its three components $\langle C_i C_i \rangle$. Moreover, the model predicts nonequal normal pressures in the three spatial directions [59], which are connected with the anisotropy of the velocity distribution function.

APPENDIX

In this appendix some results useful for the resolution of the integrals of the term $\chi(mC_iC_j)$ [74] and to express the closure equations (Sec. VII) are reported. First, let us consider two spherical particles of diameter D and mass m (Fig. 1) among which an impact is happening; if $\mathbf{c}_1$ and $\mathbf{c}_2$ are the actual velocities of the particles 1 and 2 before the impact, $\mathbf{c}_1'$ and $\mathbf{c}_2'$ the actual velocities of the particles 1 and 2 after the impact and $\mathbf{g} \equiv \mathbf{c}_1 - \mathbf{c}_2$, $\mathbf{g}' \equiv \mathbf{c}_1' - \mathbf{c}_2'$, $\mathbf{k}$ the unit vector directed from the center of the particle 1 to the center of the particle 2, some basic relations are [47]

$$\mathbf{c}_1' = \mathbf{c}_1 - \tfrac{1}{2}(1+e)(\mathbf{g}\cdot\mathbf{k})\mathbf{k}, \tag{A1}$$

$$\mathbf{c}_2' = \mathbf{c}_1 + \tfrac{1}{2}(1+e)(\mathbf{g}\cdot\mathbf{k})\mathbf{k}, \tag{A2}$$

which can be written in terms of the fluctuant velocities $\mathbf{C}_1$, $\mathbf{C}_2$ (before the impact), $\mathbf{C}_1'$, $\mathbf{C}_2'$ (after impact),

$$\mathbf{C}_1' = \mathbf{C}_1 - \tfrac{1}{2}(1+e)(\mathbf{g}\cdot\mathbf{k})\mathbf{k}, \tag{A3}$$

$$\mathbf{C}_2' = \mathbf{C}_2 + \tfrac{1}{2}(1+e)(\mathbf{g}\cdot\mathbf{k})\mathbf{k}. \tag{A4}$$

From these equations one can obtain that

$$C_{1i}'C_{1j}' - C_{1i}C_{1j} = \tfrac{1}{2}(1+e)(\mathbf{g}\cdot\mathbf{k})[\tfrac{1}{2}(1+e) \times(\mathbf{g}\cdot\mathbf{k})k_ik_j - (k_iC_{1j} + k_jC_{1i})], \tag{A5}$$

$$C_{2i}'C_{2j}' - C_{2i}C_{2j} = \tfrac{1}{2}(1+e)(\mathbf{g}\cdot\mathbf{k})[\tfrac{1}{2}(1+e) \times(\mathbf{g}\cdot\mathbf{k})k_ik_j + (k_iC_{2j} + k_jC_{2i})]. \tag{A6}$$

The variation during an impact of the product of the fluctuant velocities of the two particles $\Delta(C_iC_j)$ is given by the sum of Eqs. (A5) and (A6),

$$\Delta(C_iC_j) = \{(1+e)(\mathbf{g}\cdot\mathbf{k})k_ik_j - \lfloor k_i(g_j - u_{1j} + u_{2j})\rfloor - \lfloor k_j(g_i - u_{1i} + u_{2i})\rfloor\}\tfrac{1}{2}(1+e)(\mathbf{g}\cdot\mathbf{k}). \tag{A7}$$

A Taylor expansion truncated to the first term has been used in the evaluation of $\chi(mC_iC_j)$ to express the difference in the mean velocity components, namely,

$$u_{2j} - u_{1j} = \left(\frac{\partial u_j}{\partial r_k} k_k\right) D,$$

and similarly for i component.

Besides, the following results have been used for the expressions of the closure equations (Sec. VII). With $Y = (c_i/\sqrt{2T} - u_i/\sqrt{2T})$,

$$\int_0^{+\infty} e^{-(-u_i/\sqrt{2T} + c_i/\sqrt{2T})^2} dc_i = \sqrt{2T}\int_{-u_i/\sqrt{2T}}^{+\infty} e^{-Y^2} dY,$$

$$\int_0^{+\infty} c_i e^{-(-u_i/\sqrt{2T} + c_i/\sqrt{2T})^2} dc_i = u_i\sqrt{2T}\int_{u_i/\sqrt{2T}}^{+\infty} e^{-Y^2} dY + 2T\int_{-u_i/\sqrt{2T}}^{+\infty} e^{-Y^2} dY,$$

$$\int_0^{+\infty} c_i^2 e^{-(-u_i/\sqrt{2T} + c_i/\sqrt{2T})^2} dc_i = u_i^2\sqrt{2T}\int_{-u_i/\sqrt{2T}}^{+\infty} e^{-Y^2} dY + 4Tu_i\int_{-u_i/\sqrt{2T}}^{+\infty} Ye^{-Y^2} dY + (T2)^{3/2}\int_{-u_i/\sqrt{2T}}^{+\infty} Y^2 e^{-Y^2} dY$$

and, if $u_i = 0$,

$$\int_0^{+\infty} e^{-(-u_i/\sqrt{2T} + c_i/\sqrt{2T})^2} dc_i = \frac{1}{2}(2\pi T)^{1/2},$$

$$\int_0^{+\infty} c_i e^{-(-u_i/\sqrt{2T} + c_i/\sqrt{2T})^2} dc_i = T,$$

$$\int_0^{+\infty} c_i^2 e^{-(-u_i/\sqrt{2T} + c_i/\sqrt{2T})^2} dc_i = \sqrt{\pi 2T}\frac{T}{2}.$$

[1] R. A. Bagnold, Proc. R. Soc. London, Ser. A **255**, 49 (1954).
[2] J. T. Jenkins and S. B. Savage, J. Fluid Mech. **130**, 187 (1983).
[3] V. V. Sokolovskij, *Statics of Granular Media* (Pergamon, Oxford, 1965).
[4] D. C. Drucker and W. Prager, Q. Appl. Math. **10**, 157 (1952).
[5] G. Mandl and R. F. Luque, Geotechnique **20**, 277 (1970).
[6] M. Sayed and S. B. Savage, J. Appl. Math. Phys. **34**, 84 (1983).
[7] P. C. Johnson, P. Nott, and R. Jackson, J. Fluid Mech. **210**, 501 (1990).
[8] P. C. Johnson and R. Jackson, J. Fluid Mech. **176**, 67 (1987).
[9] S. C. Cowin, Powder Technol. **9**, 61 (1974).
[10] S. C. Cowin, Acta Mech. **20**, 41 (1974).
[11] S. C. Cowin and M. A. Goodman, Z. Angew. Math. Mech. **56**, 281 (1976).
[12] M. A. Goodman, Ph.D. thesis, Tulane University, 1970 (unpublished).
[13] M. A. Goodman and S. C. Cowin, J. Fluid Mech. **45**, 321 (1971).
[14] M. A. Goodman and S. C. Cowin, Arch. Ration. Mech. Anal. **44**, 249 (1972).
[15] J. T. Jenkins, J. Appl. Mech. **42**, 603 (1975).
[16] D. Coleman and V. J. Mizel, J. Chem. Phys. **40**, 1116 (1966).
[17] J. Serrin, J. Math. Mech. **8**, 459 (1959).
[18] S. C. Cowin, Ph.D. thesis, The Pennsylvania State University, 1962 (unpublished).
[19] S. C. Cowin, Phys. Fluids **11**, 1919 (1968).
[20] K. Kanatani, Int. J. Eng. Sci. **17**, 419 (1979).
[21] A. C. Eringen, *Nonlinear Theory of Continuous Media* (McGraw-Hill, New York, 1962).
[22] A. C. Eringen, Int. J. Eng. Sci. **2**, 205 (1964).
[23] A. C. Eringen and E. S. Suhubi, Int. J. Eng. Sci. **2**, 189 (1964).
[24] A. C. Eringen and C. B. Kafadar, in *Continuum Physics IV*, edited by A. C. Eringen (Academic Press, New York, 1976).
[25] R. D. Mindlin, Arch. Ration. Mech. Anal. **16**, 51 (1963).
[26] T. Ariman, A. S. Cakmak, and L. R. Hill, Phys. Fluids **10**, 2545 (1967).
[27] G. Capriz, *Continua with Microstructure* (Springer-Verlag, New York, 1989).
[28] G. Grioli, 1960, Ann. Mat. Pura Appl. Ser. IV **50**, 389 (1960).
[29] D. W. Condiff and J. S. Dahler, Phys. Fluids **7**, 842 (1964).
[30] D. W. Condiff, W. K. Lu, and J. S. Dahler, J. Chem. Phys. **42**, 3445 (1965).
[31] R. D. Mindlin and H. F. Tiersten, Arch. Ration. Mech. Anal. **11**, 415 (1962).
[32] R. Tiffen and A. C. Stevenson, Q. J. Mech. Appl. Math. **9**, 306 (1956).
[33] R. A. Toupin, Arch. Ration. Mech. Anal. **11**, 385 (1962).
[34] V. K. Stokes, Phys. Fluids **9**, 1709 (1966).
[35] O. Pouliquen and R. Gutfraind, Phys. Rev. E **53**, 552 (1996).
[36] A. C. Santomaso and P. Canu, Chem. Eng. Sci. **56**, 3563 (2001).
[37] C. S. Campbell, J. Fluid Mech. **203**, 449 (1989).
[38] D. M. Hanes and O. R. Walton, Powder Technol. **109**, 133 (2000).
[39] S. Ogawa, A. Umemura, and N. Oshima, Z. Angew. Math. Phys. **31**, 483 (1980).
[40] S. Ogawa, in *Continuum Mechanical and Statistical Approaches in the Mechanics of Granular Materials*, edited by S. C. Cowin and M. Satake (Gakujutsu Bunken Fukyukai, Tokyo, 1978).
[41] P. K. Haff, J. Fluid Mech. **134**, 401 (1983).
[42] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, 1970).
[43] J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes in Gases* (North-Holland, Amsterdam, 1972).
[44] F. Reif, *Fundamentals of Statistical Mechanics and Thermal Physics* (McGraw-Hill, New York, 1965).
[45] S. B. Savage and D. J. Jeffrey, J. Fluid Mech. **110**, 225 (1981).
[46] C. K. K. Lun, S. B. Savage, D. J. Jeffrey, and N. Chepurniy, J. Fluid Mech. **140**, 223 (1984).
[47] J. T. Jenkins and M. W. Richman, Arch. Ration. Mech. Anal. **87**, 355 (1985).
[48] C. K. K. Lun, J. Fluid Mech. **233**, 539 (1991).
[49] S. B. Savage and C. K. K. Lun, J. Fluid Mech. **189**, 311 (1988).
[50] S. B. Savage, J. Fluid Mech. **92**, 53 (1979).
[51] K. G. Anderson and R. Jackson, J. Fluid Mech. **241**, 145 (1992).
[52] M. W. Richman and R. P. Marciniec, J. Appl. Mech. **57**, 1036 (1990).
[53] M. Massoudi and E. J. Boyle, Int. J. Non-Linear Mech. **36**, 637 (2001).
[54] S. Warr, J. M. Huntley, and G. T. H. Jacques, Phys. Rev. E **52**, 5583 (1995).
[55] P. Nott and R. Jackson, J. Fluid Mech. **241**, 125 (1992).
[56] K. Hui, P. K. Haff, J. E. Ungar, and R. Jackson, J. Fluid Mech. **145**, 223 (1984).
[57] J. T. Jenkins and M. W. Richman, J. Fluid Mech. **171**, 53 (1986).
[58] C. Chou, Physica A **287**, 127 (2000).
[59] M. W. Richman, J. Rheol. **33(8)**, 1293 (1989).
[60] I. Goldhirsch and N. Sela, Phys. Rev. E **54**, 4458 (1996).
[61] N. Sela and I. Goldhirsch, J. Fluid Mech. **361**, 41 (1998).
[62] A. Goldshtein and M. Shapiro, J. Fluid Mech. **282**, 75 (1995).
[63] H. Grad, Commun. Pure Appl. Math. **2**, 331 (1949).
[64] H. M. Hulburt and S. Katz, Chem. Eng. Sci. **19**, 555 (1964).
[65] E. H. Kennard, *Kinetic Theory of Gases* (McGraw-Hill, New York, 1938).
[66] E. Azanza, F. Chevoir, and P. Moucheront, J. Fluid Mech. **400**, 199 (1999).
[67] T. G. Drake, J. Fluid Mech. **225**, 121 (1991).
[68] S. B. Savage, in *Continuum Mechanical and Statistical Approaches in the Mechanics of Granular Materials*, edited by S. C. Cowin and M. Satake (Gakujutsu Bunken Fukyukai, Tokyo, 1978).
[69] M. Ishida and T. Shirai, J. Chem. Eng. Jpn. **12**, 46 (1979).
[70] H. Ahn, C. E. Brennen, and R. H. Sabersky, J. Appl. Mech. **58**, 792 (1991).
[71] N. F. Carnahan and K. E. Starling, J. Chem. Phys. **51**, 635 (1969).
[72] C. S. Campbell and C. E. Brennen, J. Fluid Mech. **151**, 167 (1985).
[73] L. Verlet and D. Levesque, Mol. Phys. **46**, 969 (1982).
[74] See EPAPS Document No. E-PLEEE8-66-061210 for Appen-

dix B, containing the analytical results for the evaluation of the collisional source of the second-order moments of fluctuant velocities $\chi(mC_iC_j)$. A direct link to this document may be found in the online article's HTML reference section. The document may also be reached via the EPAPS homepage (http://www.aip.org/pubservs/epaps.html) or from ftp.aip.org in the directory /epaps/. See the EPAPS homepage for more information.